Torsion in skein modules

Giulio Belletti

Joint work with R. Detcherry

Université de Bourgogne

21/11/2024

What this talk is about

What this talk is about

Take M a 3-dimensional, compact, connected, oriented smooth manifold.

What this talk is about

Take M a 3-dimensional, compact, connected, oriented smooth manifold.

Idea: use knots and links (i.e. 1-dimensional submanifolds) to study M.

What this talk is about

Take M a 3-dimensional, compact, connected, oriented smooth manifold.

Idea: use knots and links (i.e. 1-dimensional submanifolds) to study M.

A *skein module* of M is an algebraic object that encodes links in M.

What this talk is about

Take M a 3-dimensional, compact, connected, oriented smooth manifold.

Idea: use knots and links (i.e. 1-dimensional submanifolds) to study M.

A *skein module* of M is an algebraic object that encodes links in M.

The main question will be: is torsion in the skein module related to interesting surfaces in M?

What this talk is about

Take M a 3-dimensional, compact, connected, oriented smooth manifold.

Idea: use knots and links (i.e. 1-dimensional submanifolds) to study M.

A *skein module* of M is an algebraic object that encodes links in M.

The main question will be: is torsion in the skein module related to interesting surfaces in M?

Plan of the talk:

1. What is a skein module?

What this talk is about

Take M a 3-dimensional, compact, connected, oriented smooth manifold.

Idea: use knots and links (i.e. 1-dimensional submanifolds) to study M.

A *skein module* of M is an algebraic object that encodes links in M.

The main question will be: is torsion in the skein module related to interesting surfaces in M?

Plan of the talk:

1. What is a skein module?
2. How is it connected to the character variety?

What this talk is about

Take M a 3-dimensional, compact, connected, oriented smooth manifold.

Idea: use knots and links (i.e. 1-dimensional submanifolds) to study M.

A *skein module* of M is an algebraic object that encodes links in M.

The main question will be: is torsion in the skein module related to interesting surfaces in M?

Plan of the talk:

1. What is a skein module?
2. How is it connected to the character variety?
3. How can we find torsion in the skein module?

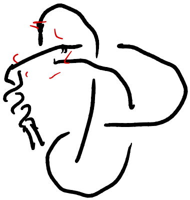
Framed Links

A *framed knot* in M is an embedding of $S^1 \times [0, \epsilon] \hookrightarrow M$.

Framed Links

A *framed knot* in M is an embedding of $S^1 \times [0, \epsilon] \hookrightarrow M$.

All the definitions for knots (links, isotopies) carry over exactly as you would imagine.



Framed Links

A *framed knot* in M is an embedding of $S^1 \times [0, \epsilon] \hookrightarrow M$.

All the definitions for knots (links, isotopies) carry over exactly as you would imagine.

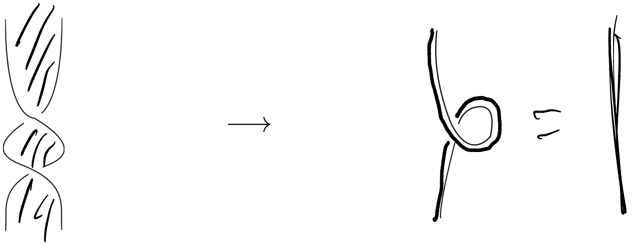
You can always project the framed link so that it lies flat on the projecting plane

Framed Links

A *framed knot* in M is an embedding of $S^1 \times [0, \epsilon] \hookrightarrow M$.

All the definitions for knots (links, isotopies) carry over exactly as you would imagine.

You can always project the framed link so that it lies flat on the projecting plane:

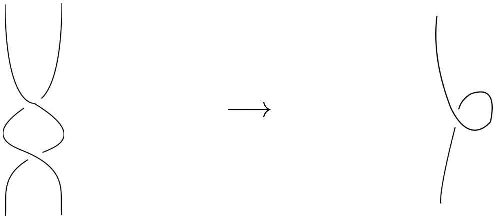


Framed Links

A *framed knot* in M is an embedding of $S^1 \times [0, \epsilon] \hookrightarrow M$.

All the definitions for knots (links, isotopies) carry over exactly as you would imagine.

You can always project the framed link so that it lies flat on the projecting plane:



Two framed links are isotopic if and only if their diagrams are connected by Reidemeister moves 2 and 3.

$\mathbb{R}^3$, S^3

Kauffman bracket skein module

1990

Definition (Przytycki, Turaev)

The *Kauffman bracket skein module* of M is defined as

$$S(M) := \mathbb{Z}[A, A^{-1}]\langle\{\text{framed links up to isotopy } L \subseteq M\}\rangle / \text{K1 and K2}$$

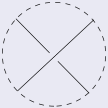
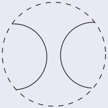
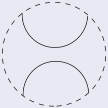
Made with Goodnotes

Kauffman bracket skein module

Definition (Przytycki, Turaev)

The *Kauffman bracket skein module* of M is defined as

$$S(M) := \mathbb{Z}[A, A^{-1}]\langle\{\text{framed links up to isotopy } L \subseteq M\}\rangle / \text{K1 and K2}$$

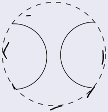
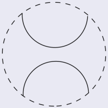
$K1 =$  $- A$  $- A^{-1}$ 

Kauffman bracket skein module

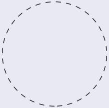
Definition (Przytycki, Turaev)

The *Kauffman bracket skein module* of M is defined as

$$S(M) := \mathbb{Z}[A, A^{-1}]\langle\{\text{framed links up to isotopy } L \subseteq M\}\rangle / \text{K1 and K2}$$

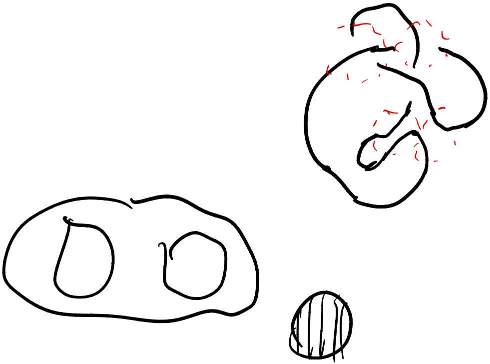
$K1 =$  $- A$  $- A^{-1}$ 

and

$K2 =$  $+ (A^2 + A^{-2})$ 

The skein module of S^3

Take any link in S^3 and project it:



The skein module of S^3

Take any link in S^3 and project it:

Take any link in S^3 and project it: you can resolve all crossings with the move K1 to obtain a linear combination of trivial curves,

The skein module of S^3

Take any link in S^3 and project it: you can resolve all crossings with the move K1 to obtain a linear combination of trivial curves, and then K2 to remove them.

The skein module of S^3

Take any link in S^3 and project it: you can resolve all crossings with the move K1 to obtain a linear combination of trivial curves, and then K2 to remove them. So any element in $S(S^3)$ is a multiple of the empty link.

The skein module of S^3

Take any link in S^3 and project it: you can resolve all crossings with the move K1 to obtain a linear combination of trivial curves, and then K2 to remove them. So any element in $S(S^3)$ is a multiple of the empty link.

$$S(S^3) \cong \mathbb{Z}[A, A^{-1}]$$

Because $S(S^3) \cong \mathbb{Z}[A, A^{-1}]$, there is a map sending any framed link L to its image under this identification: the result is *the Kauffman bracket* of L, which is a variation of the Jones polynomial.

The skein module of $S^1 \times S^2$

1995

Theorem (Hoste-Przytycki)

The skein module of $S^1 \times S^2$ is

$$S(S^1 \times S^2) = \mathbb{Z}[A, A^{-1}]$$

The skein module of $S^1 \times S^2$

Theorem (Hoste-Przytycki)

The skein module of $S^1 \times S^2$ is

$$S(S^1 \times S^2) = \mathbb{Z}[A, A^{-1}] \oplus$$

The skein module of $S^1 \times S^2$

Theorem (Hoste-Przytycki)

The skein module of $S^1 \times S^2$ is

$$S(S^1 \times S^2) = \mathbb{Z}[A, A^{-1}] \oplus \bigoplus_{i=0}^{\infty} \mathbb{Z}[A, A^{-1}]/(1 - A^{2i+4})$$

The skein module of $S^1 \times S^2$

Theorem (Hoste-Przytycki)

The skein module of $S^1 \times S^2$ is

$$S(S^1 \times S^2) = \mathbb{Z}[A, A^{-1}] \oplus \bigoplus_{i=0}^{\infty} \mathbb{Z}[A, A^{-1}]/(1 - A^{2i+4})$$

Easy to see that $S(S^1 \times S^2)$ should have torsion.

The skein module of $S^1 \times S^2$

Theorem (Hoste-Przytycki)

The skein module of $S^1 \times S^2$ is

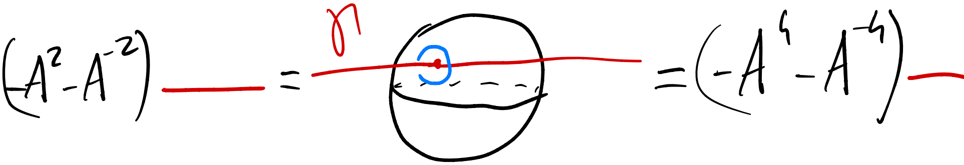
$$S(S^1 \times S^2) = \mathbb{Z}[A, A^{-1}] \oplus \bigoplus_{i=0}^{\infty} \mathbb{Z}[A, A^{-1}]/(1 - A^{2i+4})$$

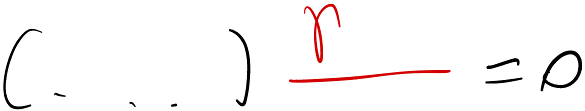
Easy to see that $S(S^1 \times S^2)$ should have torsion.

$S(S^1 \times D^2) = \mathbb{Z}[A, A^{-1}][x]$

$S^1 \times \Sigma_g$

A torsion element in $S^1 \times S^2$





Idea: $\varphi \quad S(M) \longrightarrow V$

$\varphi(\gamma) \neq 0$

Made with Goodnotes

The $SL(2,\mathbb{C})$-character variety of a manifold

$SL(2,\mathbb{C})^n$

$\cup //$

$$\{\text{Homomorphisms } \pi_1(M) \to SL(2,\mathbb{C})\}$$

The $SL(2,\mathbb{C})$-character variety of a manifold

$$\{\text{Homomorphisms } \pi_1(M) \to SL(2,\mathbb{C})\}//\text{conjugation}$$

The $SL(2,\mathbb{C})$-character variety of a manifold

$$\chi(M) = \{\text{Homomorphisms } \pi_1(M) \to SL(2,\mathbb{C})\}//\text{conjugation}$$

Made with Goodnotes

The $SL(2,\mathbb{C})$-character variety of a manifold

$$\chi(M) = \{\text{Homomorphisms } \pi_1(M) \to SL(2,\mathbb{C})\}//\text{conjugation}$$

The GIT quotient is needed because the orbits of the conjugation action in $SL(2,\mathbb{C})$ are not closed; this quotient takes as points closure of orbits.

$$\chi(M) = \{\text{Homomorphisms } \pi_1(M) \to SL(2,\mathbb{C})\}//\text{conjugation}$$

The GIT quotient is needed because the orbits of the conjugation action in $SL(2,\mathbb{C})$ are not closed; this quotient takes as points closure of orbits. This is an affine algebraic variety, and its algebra of functions is generated by traces:

$$\mathrm{tr}_\gamma$$

$$\uparrow$$

$$\pi_1(M)$$

The $SL(2,\mathbb{C})$-character variety of a manifold

$$\chi(M) = \{\text{Homomorphisms } \pi_1(M) \to SL(2,\mathbb{C})\}//\text{conjugation}$$

The GIT quotient is needed because the orbits of the conjugation action in $SL(2,\mathbb{C})$ are not closed; this quotient takes as points closure of orbits. This is an affine algebraic variety, and its algebra of functions is generated by traces:

$$\mathrm{tr}_\gamma(\rho)$$

The $SL(2,\mathbb{C})$-character variety of a manifold

$$\chi(M) = \{\text{Homomorphisms } \pi_1(M) \to SL(2,\mathbb{C})\}//\text{conjugation}$$

The GIT quotient is needed because the orbits of the conjugation action in $SL(2,\mathbb{C})$ are not closed; this quotient takes as points closure of orbits.
This is an affine algebraic variety, and its algebra of functions is generated by traces:

$$\mathrm{tr}_\gamma(\rho) = \mathrm{tr}(\rho(\gamma))$$

Example: character variety of $S^1 \times S^2$

In this case $\pi_1(S^1 \times S^2) = \mathbb{Z}$, so any representation ρ is determined by $\rho(\gamma)$, with γ any generator for $\mathbb{Z}$ (for example, the loop $S^1 \times \{*\}$).

In this case $\pi_1(S^1 \times S^2) = \mathbb{Z}$, so any representation ρ is determined by $\rho(\gamma)$, with γ any generator for $\mathbb{Z}$ (for example, the loop $S^1 \times \{*\}$).

If $\rho(\gamma)$ is diagonalizable, then it is conjugate to $\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$. $\simeq \begin{pmatrix} z^{-1} & 0 \\ 0 & z \end{pmatrix}$

Example: character variety of $S^1 \times S^2$

In this case $\pi_1(S^1 \times S^2) = \mathbb{Z}$, so any representation ρ is determined by $\rho(\gamma)$, with γ any generator for $\mathbb{Z}$ (for example, the loop $S^1 \times \{*\}$).

If $\rho(\gamma)$ is diagonalizable, then it is conjugate to $\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$. ,

If $\rho(\gamma)$ is not diagonalizable, then it is conjugate to $\pm \begin{pmatrix} 1 & \epsilon \\ 0 & 1 \end{pmatrix}$ for any

$\epsilon \neq 0$.

Example: character variety of $S^1 \times S^2$

In this case $\pi_1(S^1 \times S^2) = \mathbb{Z}$, so any representation ρ is determined by $\rho(\gamma)$, with γ any generator for $\mathbb{Z}$ (for example, the loop $S^1 \times \{*\}$).

If $\rho(\gamma)$ is diagonalizable, then it is conjugate to $\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$.

If $\rho(\gamma)$ is not diagonalizable, then it is conjugate to $\pm \begin{pmatrix} 1 & \epsilon \\ 0 & 1 \end{pmatrix}$ for any $\epsilon \neq 0$. Therefore the closure of this orbit contains $\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Example: character variety of $S^1 \times S^2$

In this case $\pi_1(S^1 \times S^2) = \mathbb{Z}$, so any representation ρ is determined by $\rho(\gamma)$, with γ any generator for $\mathbb{Z}$ (for example, the loop $S^1 \times \{*\}$).

If $\rho(\gamma)$ is diagonalizable, then it is conjugate to $\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$.

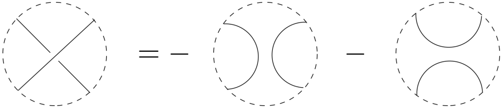
If $\rho(\gamma)$ is not diagonalizable, then it is conjugate to $\pm \begin{pmatrix} 1 & \epsilon \\ 0 & 1 \end{pmatrix}$ for any $\epsilon \neq 0$. Therefore the closure of this orbit contains $\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

All this says that $\chi(S^1 \times S^2) = \mathbb{C}^*/\tau$, where $\tau(x) = x^{-1}$.

$tr_\rho(x) = x + x^{-1}$

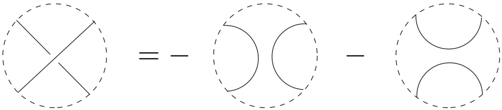
The skein module at -1

Look at the Kauffman relations at -1:



The skein module at -1

Look at the Kauffman relations at -1:

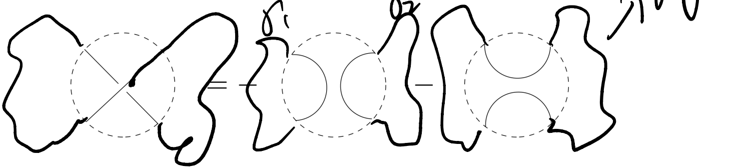


A bit of work shows that it looks like the Cayley-Hamilton equation

$$\mathrm{tr}(A)\mathrm{tr}(B) = \mathrm{tr}(AB) + \mathrm{tr}(AB^{-1})$$

The skein module at -1

Look at the Kauffman relations at -1:



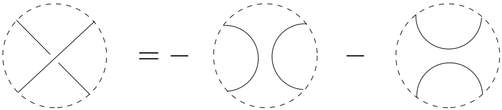
A bit of work shows that it looks like the Cayley-Hamilton equation

$$\mathrm{tr}(A)\mathrm{tr}(B) = \mathrm{tr}(AB) + \mathrm{tr}(AB^{-1})$$

Sending a link $L = L_1 \sqcup \cdots \sqcup L_n$ to $\prod_1^n(-\mathrm{tr}_{L_i})$ and A to -1 gives a map between $S(M)$ and $\mathbb{C}[\chi(M)]$ (Bullock);

The skein module at -1

Look at the Kauffman relations at -1:

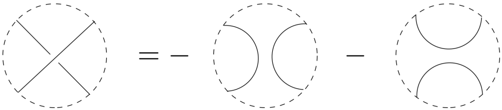


A bit of work shows that it looks like the Cayley-Hamilton equation

$$\mathrm{tr}(A)\mathrm{tr}(B) = \mathrm{tr}(AB) + \mathrm{tr}(AB^{-1})$$

Sending a link $L = L_1 \sqcup \cdots \sqcup L_n$ to $\prod_1^n(-\mathrm{tr}_{L_i})$ and A to -1 gives a map between $S(M)$ and $\mathbb{C}[\chi(M)]$ (Bullock); and actually
$S_{-1}(M)$

Look at the Kauffman relations at -1:

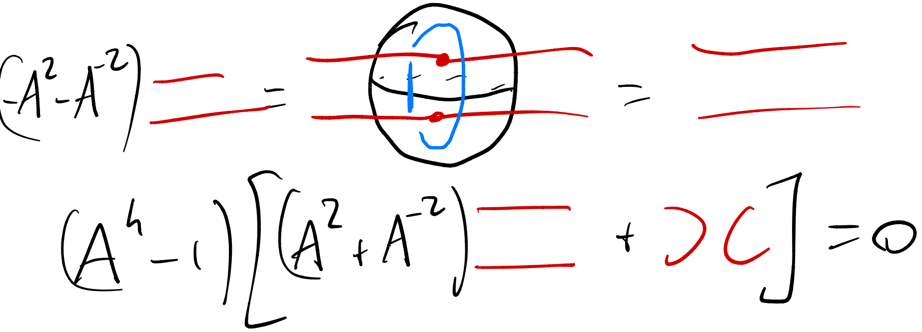


A bit of work shows that it looks like the Cayley-Hamilton equation

$$\mathrm{tr}(A)\mathrm{tr}(B) = \mathrm{tr}(AB) + \mathrm{tr}(AB^{-1})$$

Sending a link $L = L_1 \sqcup \cdots \sqcup L_n$ to $\prod_1^n (-\mathrm{tr}_{L_i})$ and A to -1 gives a map between $S(M)$ and $\mathbb{C}[\chi(M)]$ (Bullock); and actually $S_{-1}(M) = S(M) \otimes_{A=-1} \mathbb{C} = \mathbb{C}[\chi(M)]$ (Przytycki-Sikora).

More torsion elements



Connected sums

If M contains a sphere that does not bound a ball, then we can find torsion candidates.

Made with Goodnotes

Connected sums

If M contains a sphere that does not bound a ball, then we can find torsion candidates.

If M contains a separating sphere S that does not bound a ball, then we can cut M along (an open tubular neighborhood of) S,

Connected sums

If M contains a sphere that does not bound a ball, then we can find torsion candidates.

If M contains a separating sphere S that does not bound a ball, then we can cut M along (an open tubular neighborhood of) S, and fill the resulting boundary components to obtain two manifolds M_1 and M_2.

Connected sums

If M contains a sphere that does not bound a ball, then we can find torsion candidates.

$\Downarrow$

If M contains a separating sphere S that does not bound a ball, then we can cut M along (an open tubular neighborhood of) S, and fill the resulting boundary components to obtain two manifolds M_1 and M_2.

Viceversa, if we have M_1 and M_2 we can remove two balls from them and glue them along the resulting boundary components to obtain their *connected sum* $M_1 \# M_2$.

Connected sums

If M contains a sphere that does not bound a ball, then we can find torsion candidates.

If M contains a separating sphere S that does not bound a ball, then we can cut M along (an open tubular neighborhood of) S, and fill the resulting boundary components to obtain two manifolds M_1 and M_2.

Viceversa, if we have M_1 and M_2 we can remove two balls from them and glue them along the resulting boundary components to obtain their *connected sum* $M_1 \# M_2$.

If M contains a non-separating sphere, actually $M = M' \# S^1 \times S^2$.

A manifold like $M_1 \# M_2$ is reducible (irreducible otherwise)

Torsion in skein modules, I

1997

Conjecture (Przytycki)

Let M be a compact oriented reducible 3-manifold; then $S(M)$ contains torsion.

Made with Goodnotes

Essential surfaces

An *essential sphere* in M is an embedded sphere that does not bound a ball.

Essential surfaces

An *essential sphere* in M is an embedded sphere that does not bound a ball.

An *essential surface* in M is an (orientable) surface embedded in M such that the inclusion map induces an injective map on π_1s.

Torsion in skein modules, I

Conjecture (Przytycki)

Let M be a compact oriented reducible 3-manifold; then $S(M)$ contains torsion.

Torsion in skein modules, I

Conjecture (Przytycki)

Let M be a compact oriented reducible 3-manifold; then $S(M)$ contains torsion.

Conjecture (Przytycki)

Let M be a compact oriented 3-manifold that does not contain any essential, non-boundary parallel closed surface; then $S(M)$ is a free module (and hence torsion-free).

Conjecture (Przytycki)

Let M be a compact oriented reducible 3-manifold; then $S(M)$ contains torsion.

$M_1 \# M_2$, $M_1 \# S^1 \times S^2$

Conjecture (Przytycki)

Let M be a compact oriented 3-manifold that does not contain any essential, non-boundary parallel closed surface; then $S(M)$ is a free module (and hence torsion-free).

What happens in between? It was known at the time that sometimes essential tori give rise to torsion, but nothing was known for higher genus surfaces.

List of known results

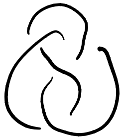
- $S(M_1 \# M_2)$ has $(A \pm 1)$-torsion if M_1 and M_2 are rational homology spheres and neither M_1 nor M_2 are connected sums of (any number of) $\mathbb{RP}^3$s (due to Przytycki, Zentner);

List of known results

- $S(M_1 \# M_2)$ has $(A \pm 1)$-torsion if M_1 and M_2 are rational homology spheres and neither M_1 nor M_2 are connected sums of (any number of) $\mathbb{RP}^3$s (due to Przytycki, Zentner);
- $\mathbb{RP}^3 \# \mathbb{RP}^3$ has $(A \pm i)$-torsion but no $(A \pm 1)$-torsion (Mroczkowski)

List of known results

- $S(M_1 \# M_2)$ has $(A \pm 1)$-torsion if M_1 and M_2 are rational homology spheres and neither M_1 nor M_2 are connected sums of (any number of) $\mathbb{RP}^3$s (due to Przytycki, Zentner);
- $\mathbb{RP}^3 \# \mathbb{RP}^3$ has $(A \pm i)$-torsion but no $(A \pm 1)$-torsion (Mroczkowski)
- The double of the figure eight knot exterior has $(A \pm 1)$-torsion (Veve).



Some new results

Some new results

J.W. RENAUD DETCHERRY

- If M is closed and $b_1(M) = \dim H_1(M, \mathbb{Q}) > 0$ then $S(M)$ contains $(A \pm 1)$-torsion.

Some new results

M has ess. surf.

$\Uparrow$

- If M is closed and $b_1(M) = \dim H_1(M, \mathbb{Q}) > 0$ then $S(M)$ contains $(A \pm 1)$-torsion.
- There is an M such that $S(M)$ has torsion but M does not contain any essential surface of genus less than g, for every g.

Some new results

- If M is closed and $b_1(M) = \dim H_1(M, \mathbb{Q}) > 0$ then $S(M)$ contains $(A \pm 1)$-torsion.
- There is an M such that $S(M)$ has torsion but M does not contain any essential surface of genus less than g, for every g.
- $S\left(L\left(p, 1\right) \#^n \mathbb{RP}^3\right)$ contains $(A^2 + 1)$-torsion for all even p and for all $n \geq 1$.

Some new results

- If M is closed and $b_1(M) = \dim H_1(M, \mathbb{Q}) > 0$ then $S(M)$ contains $(A \pm 1)$-torsion.
- There is an M such that $S(M)$ has torsion but M does not contain any essential surface of genus less than g, for every g.
- $S\left(L(p,1) \#^n \mathbb{RP}^3\right)$ contains $(A^2 + 1)$-torsion for all even p and for all $n \geq 1$.
- If M is a *Seifert manifold*

Some new results

- If M is closed and $b_1(M) = \dim H_1(M, \mathbb{Q}) > 0$ then $S(M)$ contains $(A \pm 1)$-torsion.
- There is an M such that $S(M)$ has torsion but M does not contain any essential surface of genus less than g, for every g.
- $S\left(L(p,1) \#^n \mathbb{RP}^3\right)$ contains (A^2+1)-torsion for all even p and for all $n \geq 1$.
- If M is a *Seifert manifold* that contains an essential closed surface not parallel to the boundary, then $S(M)$ has $A \pm 1$ torsion.

Torsion in skein modules, II

Conjecture (Przytycki)

Let M be a compact oriented non-irreducible 3-manifold; then $S(M)$ contains torsion.

Conjecture (Przytycki)

Let M be a compact oriented 3-manifold that does not contain any essential, non-boundary parallel surface; then $S(M)$ is a free module (and hence torsion-free).

What happens in between? It was known at the time that sometimes essential tori give rise to torsion, but nothing was known for higher genus surfaces.

Torsion in skein modules, II

Conjecture (Przytycki)

Let M be a compact oriented 3-manifold. Then the following are equivalent:

(1) *M contains an embedded essential, closed, surface that is not parallel to the boundary.*

(2) *$\mathcal{S}(M)$ has torsion.*

Torsion in skein modules, II

Conjecture (Przytycki)

Let M be a compact oriented 3-manifold. Then the following are equivalent:

(1) *M contains an embedded essential, closed, surface that is not parallel to the boundary.*

(2) *$\mathcal{S}(M)$ has torsion.* ⇐

If M is closed, both are equivalent to

(3) *$S(M)$ is not finitely generated over $\mathbb{Z}[A, A^{-1}]$* ⇐
(Detcherry-Kalfagianni-Sikora). ⇐

$S(M) = \mathbb{Z}[A, A^{-1}]$?

$S^1 \times S^2$, $\mathbb{RP}^3 \# \mathbb{RP}^3$, $S^1 \times S^2 \# S^1 \times S^2$

BAKSHI, KIM, SHI, WANG 24

Made with Goodnotes

From $b_1 > 0$ to torsion in the skein module

From $b_1 > 0$ to torsion in the skein module

Theorem (Gunningham-Jordan-Safranov)

For a closed manifold M, the skein module $S(M, \mathbb{Q}(A))$ is finite dimensional.

From $b_1 > 0$ to torsion in the skein module

Theorem (Gunningham-Jordan-Safranov)

For a closed manifold M, the skein module $S(M, \mathbb{Q}(A))$ is finite dimensional.

If $b_1(M)$ is positive, then $\mathbb{C}[\chi(M)]$ is an infinite dimensional vector space.

From $b_1 > 0$ to torsion in the skein module

Theorem (Gunningham-Jordan-Safranov)

For a closed manifold M, the skein module $S(M, \mathbb{Q}(A))$ is finite dimensional.

If $b_1(M)$ is positive, then $\mathbb{C}[\chi(M)]$ is an infinite dimensional vector space. Find linearly independent $\lambda_1, \ldots, \lambda_n, \cdots \in \mathbb{C}[\chi(M)]$ and look at their preimage in $S(M)$.

From $b_1 > 0$ to torsion in the skein module

Theorem (Gunningham-Jordan-Safranov)

For a closed manifold M, the skein module $S(M, \mathbb{Q}(A))$ is finite dimensional.

If $b_1(M)$ is positive, then $\mathbb{C}[\chi(M)]$ is an infinite dimensional vector space. Find linearly independent $\lambda_1, \ldots, \lambda_n, \cdots \in \mathbb{C}[\chi(M)]$ and look at their preimage in $S(M)$.

These elements must be linearly dependent in $S(M, \mathbb{Q}(A))$: $\sum_i P_i(A)\lambda'_i = 0$.

From $b_1 > 0$ to torsion in the skein module

Theorem (Gunningham-Jordan-Safranov)

For a closed manifold M, the skein module $S(M, \mathbb{Q}(A))$ is finite dimensional.

If $b_1(M)$ is positive, then $\mathbb{C}[\chi(M)]$ is an infinite dimensional vector space. Find linearly independent $\lambda_1, \dots, \lambda_n, \dots \in \mathbb{C}[\chi(M)]$ and look at their preimage in $S(M)$.

These elements must be linearly dependent in $S(M, \mathbb{Q}(A))$: $\sum_i P_i(A)\lambda_i' = 0$.

However they are linearly independent in $\mathbb{C}[\chi(M)]$, which means $P_i(-1) = 0$ for all is.

From $b_1 > 0$ to torsion in the skein module

Theorem (Gunningham-Jordan-Safranov)

For a closed manifold M, the skein module $S(M, \mathbb{Q}(A))$ is finite dimensional.

If $b_1(M)$ is positive, then $\mathbb{C}[\chi(M)]$ is an infinite dimensional vector space. Find linearly independent $\lambda_1, \ldots, \lambda_n, \cdots \in \mathbb{C}[\chi(M)]$ and look at their preimage in $S(M)$.

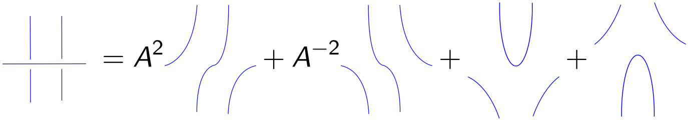
These elements must be linearly dependent in $S(M, \mathbb{Q}(A))$: $\sum_i P_i(A)\lambda_i' = 0$.

However they are linearly independent in $\mathbb{C}[\chi(M)]$, which means $P_i(-1) = 0$ for all is.

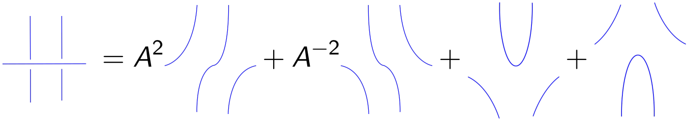
This means $(A+1)^k \left(\sum_i P_i'(A)\lambda_i'\right) = 0$.

Using a torus to find torsion

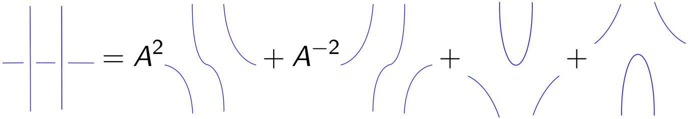
Using a torus to find torsion

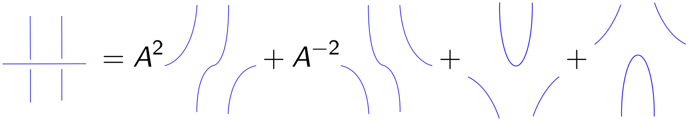


Using a torus to find torsion

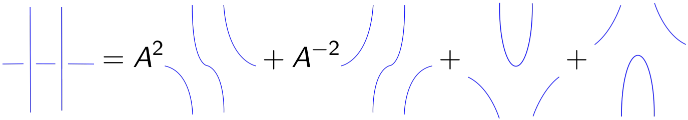


and

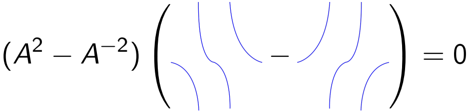




and



Therefore



The main criterion

Take M that contains an essential, separating, non-boundary parallel torus T. Then cutting M along T produces two manifolds M_1 and M_2 with some toric boundary.

The main criterion

Take M that contains an essential, separating, non-boundary parallel torus T. Then cutting M along T produces two manifolds M_1 and M_2 with some toric boundary.

Theorem

Let M_1, M_2, M, T be as above. Suppose that $\rho : \pi_1(M) \to SL(2, \mathbb{C})$ is a representation satisfying the following:

- *ρ is irreducible;*
- *ρ restricts to non-abelian representations of $\pi_1(M_1)$ and $\pi_1(M_2)$;*
- *ρ restricts to a non-central representation of $\pi_1(T) \subseteq \pi_1(M)$.*

Then $\mathcal{S}(M)$ contains $(A \pm 1)$-torsion.

The main criterion

Take M that contains an essential, separating, non-boundary parallel torus T. Then cutting M along T produces two manifolds M_1 and M_2 with some toric boundary.

Theorem

Let M_1, M_2, M, T be as above. Suppose that $\rho : \pi_1(M) \to SL(2, \mathbb{C})$ is a representation satisfying the following:

- *ρ is irreducible;*
- *ρ restricts to non-abelian representations of $\pi_1(M_1)$ and $\pi_1(M_2)$;*
- *ρ restricts to a non-central representation of $\pi_1(T) \subseteq \pi_1(M)$.*

Then $\mathcal{S}(M)$ contains $(A \pm 1)$-torsion.

There is also a similar criterion for non-separating tori but it is less clean.

Further problems

- Finding concrete torsion elements arising from higher genus surfaces;

Further problems

- Finding concrete torsion elements arising from higher genus surfaces;
- defining maps $S_\zeta(M)$ for some roots of unity, to verify $A - \zeta$ torsion elements;

Further problems

- Finding concrete torsion elements arising from higher genus surfaces;
- defining maps $S_\zeta(M)$ for some roots of unity, to verify $A - \zeta$ torsion elements;
- What is the minimal torsion you can get?

Thm DKS

Under conditions, $S(M) = \mathbb{Z}[A, A^{-1}]^n$

where $n = |X(M)|$

$\mathbb{Z}[A, A^{-1}]$

$S(M, \mathbb{Q}(A))$ is f.d. for closed M.

$\uparrow$

12/12 LIU

Made with Goodnotes