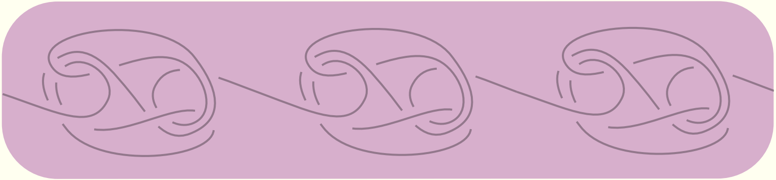
Involutions on S^4



Keegan Boyle
(joint work with Wenzhao Chen and Anthony Conway)

April 16, 2026

Motivating Question

Question

What are the (non-identity) $\mathbb{Z}/2$ actions (involutions) on S^n?

Definition

Two involutions $\rho\colon S^n \to S^n$ *are (topologically)* equivalent *if they are conjugate in* $Homeo(S^n)$.

The 2-sphere

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Remark

Smooth involutions are locally linear.

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Let F be the fixed-point set. Then we have linear involutions:

- ($\dim(F) = -1$) The antipodal involution (with quotient $\mathbb{R}P^4$),
- ($\dim(F) = 0$) ‘Rotation’ around an S^0,
- ($\dim(F) = 1$) ‘Reflection’ across an S^1,
- ($\dim(F) = 2$) Rotation around an S^2, and
- ($\dim(F) = 3$) Reflection across an S^3.

Fixed-point set is empty or 0-dimensionl

Theorem (Hambleton-Kreck-Teichner 1994)

There are exactly two free locally linear involuions on S^4*, distinguished by the Kirby-Siebenmann invariants of their quotients.*

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Theorem (Gordon 1973, Sumners 1973)

There are infinitely many distinct locally linear involutions on S^4 *with fixed-point set* $F \cong S^2$, *distinguished by* $\pi_1(S^4 - F)$.

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Theorem

If ρ is a locally linear involution on S^4 with $\pi_1(S^4 - F) \cong \mathbb{Z}$, then ρ is linear.

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Theorem (Corollary of Freedman 1982)

There is a bijection between integer homology 3-spheres, and locally linear involutions on S^4 *with a 3-dimensional fixed-point set.*

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A $\mathbb{Z}HS^3$ Y bounds a unique contractible 4-manifold, and doubling this gives S^4. Conversely, by Smith theory, the fixed-point set of such an involution is an $\mathbb{Z}HS^3$. ☐

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Corollary

Every smooth involution on S^4 *with a 1-dimensional fixed-point set is topologically conjugate to a linear involution.*

Proof idea

- Cut out an equivariant tubular neighborhood of F to get $X = S^4 - F \cong S^2 \times D^2$.

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Proof: Follow the classification of closed topological 4-manifolds with $\pi_1 = \mathbb{Z}/2$ [Hambleton-Kreck-Teichner 1994].

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In particular, taking the double covers of $\overline{X}_0$ and $\overline{X}_1$, then gluing back in a symmetric $S^1 \times D^3$ gives the linear action on S^4 in both cases!

Non-uniqueness theorem

Theorem (B.-Chen-Conway 2025)

Let ρ be a fixed-point linear involution on S^4 with a 1-dimensional fixed-point set. Then $F = \mathsf{Fix}(\rho)$ has exactly two equivariant tubular neighborhoods up to equivariant homeomorphism of pairs $(S^4, \nu(F))$.

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It is worth comparing this to the non-equivariant setting:

Theorem (Freedman-Quinn 1990, Theorem 9.3)

Let A be a locally flat submanifold in a 4-manifold M. Then A has a unique tubular neighborhood up to ambient isotopy, and in particular up to homeomorphism of pairs $(M, \nu(A))$.

Equivariant tubular neighborhoods

Proposition

If $\dim(F) \neq 1$, *then any locally linear involution on* S^4 *is fixed-point linear, and in fact the equivariant tubular neighborhood* $\nu(F)$ *is unique.*

Question (Non-existence of ETNs)

Is there a locally linear involution on S^4 which is not fixed-point linear?

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By the Proposition, such an involution would have 1-dimensional fixed-point set.

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How do we know that $\overline{X}_0 \cong \mathbb{R}P^2 \times D^2$ and $\overline{X}_1$ produce different ETNs and not different involutions?

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Theorem (Equivariant Schoenflies Theorem)

Let $\rho\colon S^4 \to S^4$ *be a linear involution and let* $\rho_3\colon S^3 \to S^3$ *be a linear involution with* $\dim(F_\rho) - 1 = \dim(F_{\rho_3})$.

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Proof.

The Bing-shrinking argument works equivariantly. ☐

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Theorem

There is a unique fixed-point linear involution ρ on S^4 with $\dim(F = \mathit{Fix}(\rho)) = 1$*. (I.e. all such involutions are linear.)*

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□

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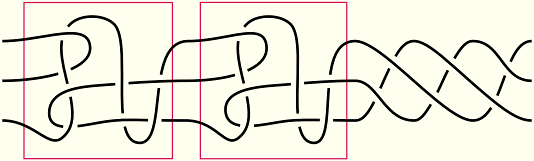

Figure: The closure of this braid is a freely 2-periodic knot.

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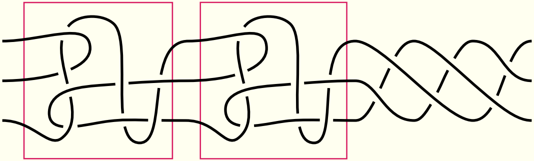

Figure: The closure of this braid is a freely 2-periodic knot.

In this case, an extenion of ρ_3 to B^4 will have a 0-dimensional fixed-point set, so it is unique by [Kwasik-Schultz 1989].

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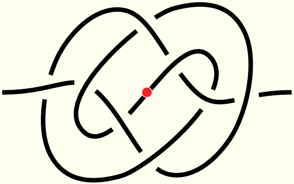

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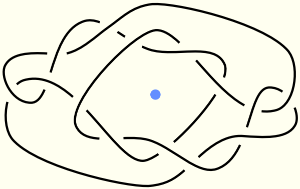

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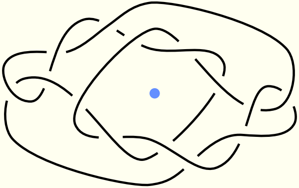

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In these cases, a fixed-point linear extension of ρ_3 to B^4 will have a 1-dimensional fixed point set, so it is unique by [B.-Chen-Conway 2025].

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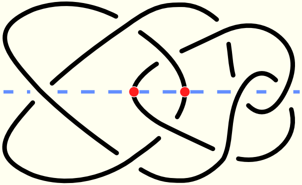

Figure: A strongly invertible knot.

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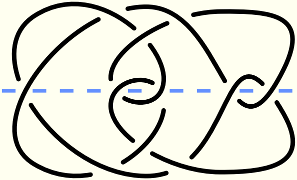

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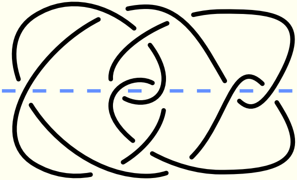

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- If $\dim(F_{\rho_3}) = 2$, no prime knot is ρ_3-invariant.

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Example theorem:

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Definition

A ρ_3-invariant knot is standardly *equivariantly slice if it is equivariantly slice with respect to the linear extension of ρ_3 to B_4.*

Example theorem:

Theorem (Corollary of Kwasik-Schultz 1989)

If K is a freely 2-periodic knot which is equivariantly (topologically) slice, then K is standardly equivariantly (topologically) slice.

Equivariant 4-genera

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For another example, the following theorem

Theorem (B.-Chen, 2024)

Every strongly negative amphichiral knot K with $\Delta_K(t) = 1$ is equivariantly (topologically) slice (with respect to some extension of ρ_3 to B^4).

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Theorem (B.-Chen-Conway, 2025)

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Questions

What happens in the smooth category? Are there exotic involutions on S^4 with $0, 1$, or 2 dimensional fixed-point sets?

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Can you use equivariant 4-genera of symmetric knots to distinguish involutions?

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Thanks! Questions?