On Calegari's homotopy 4-spheres (joint with Min Hoon Kim)

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SPC4: Is every smooth homotopy 4-sphere diffeo. to S^4?

Sources of potential counterexamples:

- Gluck twist along $S^2 \hookrightarrow S^4$
- Andrews-Curtis Conj.
- Cappell-Shaneson spheres $\Sigma_A^\varepsilon \leftsquigarrow \left[\begin{array}{l} A \in SL(3,\mathbb{Z}) \\ \det(I-A)=\pm 1, \\ \varepsilon \in \{+1,-1\} \end{array}\right]$ parametrized by $\downarrow$ 3 ints

Calegari's construction (2009)

$K \subset S^3$ fibered, monodromy $S \xrightarrow{\cong} S$ = min. Seifert surf.

$\rightsquigarrow$ automorphism $\phi : F \xrightarrow[\cong]{} F = \langle x_1, \cdots, x_n \mid \cdot \rangle = \pi_1 S$

$n = 2 \cdot \text{genus}$

Call ϕ a geometric automorphism.

Let $M = M_n = \#^n S^1 \times S^2 \smallsetminus (\text{open 3-ball})$; $\partial M = S^2$

Choose a diffeo. $f : M \xrightarrow[\cong]{} M$ s.t. $f|_{\partial M} = id$, $f_* = \phi$ on π_1.

Such f exists, since $\mathrm{Aut}(F)$ is gen. by Nielsen trans.

Define $\Sigma(f) := \left(M \times [0,1] \big/ (f(x),0) \sim (x,1) \right) \bigcup_{\partial M \times S^1 = S^2 \times S^1} S^2 \times D^2$

"relative mapping torus".

Prop. (Calegari): $\Sigma(f)$ is a homotopy 4-sphere.

Proof. $\pi_1 \Sigma(f) = \langle x_1, \cdots, x_n, t \mid \phi(x_i) = t x_i t^{-1}, t \rangle \cong \pi_1 S^3 = \{1\}$
$i = 1, \cdots, n$

Verify $H_2(\Sigma(f); \mathbb{Z}) = 0$, using ... (P_ϕ)

Question (Calegari): Is $\Sigma(f)$ diffeo. to S^4, or not?

Remarks:

(1) $\exists$ many $\left\{ \begin{array}{l} \text{fibered knots} \\ \text{geometric automorphisms} \end{array} \right\}$, obtained systematically.

(2) A fibered knot of genus g

$\rightsquigarrow 2^n$ diffeo. $f : M \xrightarrow{\cong} M$, giving $\Sigma(f)$.
$(n = 2g)$

(3) Handle structure of $\Sigma(f)$: # of handles grows linearly on n
(Kirby diagram)

Theorem [C.-M.H. Kim 2024]

All Calegari spheres from fibered knots are diffeo. to S^4.

Strategy of proof:

(1) Use 3D mapping class group theory to express $\Sigma(f) = \partial P^5$, P^5 related to P_ϕ

(2) Use the fibered knot to construct a handle decomposition of D^3, related to P_ϕ.

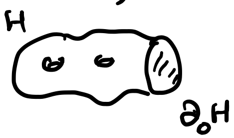
(3) Compare P^5 and $\underline{D^3 \times D^2} = D^5$:

cf. $M = \#^n S^1 \times S^2 - (\text{3-ball})$

(1): Let $H = H_n = \natural^n S^1 \times D^2$, genus n 3D handlebody.

Fix $\partial_0 H \cong D^2 \subset \partial H$ $\qquad D(H) = \#^n S^1 \times S^2$

Then:
$$M = H \underset{\partial H \setminus \partial_0 H}{\cup} -H$$
$$S^2 = \partial M = \partial_0 H \underset{\partial}{\cup} -\partial_0 H$$



$h: H \xrightarrow[\text{diffeo}]{\cong} H$ fixing $\partial_0 H$ induced $D(h) = f: M \xrightarrow{\cong} M$ fixing ∂M.

Lemma: Every $f: M \xrightarrow{\cong} M$ fixing ∂M is isotopic, rel ∂, to $D(h)$ for some h.

Sublemma: Given $\phi \in \mathrm{Aut}(F)$, $\exists$ exactly 2^n classes $[f] \in \mathrm{Mod}(M, \partial M)$ inducing ϕ on $\pi_1 M = F$

Proof: Laudenbach's exact sequence $\cong \mathrm{Mod}(M, \partial M)$.
$$0 \to (\mathbb{Z}_2)^n \to \mathrm{Mod}(\#^n \underline{S^1 \times S^2}, *) \to \mathrm{Aut}(F) \to 0$$

gen. by the n "sphere twists" along $*\times S^2$
$$S \times [0,1] \xrightarrow{\cong} S \times [0,1]$$
$$(x,t) \mapsto (R_{2\pi t}(x), t)$$
$2\pi t$-rotation.

- sphere twist has order 2

Proof of lemma

For a given $f: M \xrightarrow{\cong} M$, $\exists h_0 : H \xrightarrow{\cong} H$ s.t. $\phi = f_* = h_{0*}$

$f = D(h_0) \circ$ (composition of some of the sphere twists)

each is the double of the "disk twist" $* \times D^2 \subset \natural^n S^1 \times D^2 = H$

$\Rightarrow f = D(h)$ for some h ///

Consequence: $\Sigma(f) = D(\Sigma(h))$ for some $h: H \xrightarrow{\cong} H$

where $\Sigma(h) = \left(H \times [0,1] / \sim \right) \underset{\partial_0 H \times S^1 = D^2 \times S^1}{\cup} D^2 \times D^2$

"Casson-Gordon contractible 4-ball"

$\partial \Sigma(h) = \mathbb{Z}HS^3$ but $\neq S^3$ in general.

Let $P^5 := \Sigma(h) \times I$; $\partial P = D(\Sigma(h)) = \Sigma(f)$.

$= (0\text{-handle}) \cup (\underbrace{n \text{ 1-handle}}) \cup (\underbrace{1\text{-handle}}) \cup (n+1 \text{ 2-handles})$

cf. $\pi_1 \Sigma(h) = \pi_1 P^5 = \underline{\langle x_1, \cdots, x_n, t \mid \phi(x_i) = t x_i t^{-1}, t \rangle}$

(2) Recall $\phi \in \mathrm{Aut}(F)$ induced by monodromy $h_0 : S \xrightarrow{\cong} S$ of the fibered knot $K \subset S^3$:

$D^3 = \big(\underbrace{S \times [0,1] / \sim}_{\text{mapping torus}}\big) \underset{D^1 \times S^1}{\cup} D^1 \times D^2$

$= (0\text{-handle}) \cup (n+1 \text{ 1-handles}) \cup (n+1 \text{ 2-handles})$

(3) Compare P and $D^5 = D^3 \times D^2$

a-circles of 2-handles are homotopic $\Rightarrow$ isotopy

framing issue:

∂P^5 and $\partial D^5 = S^4$ are related by Gluck twists along $\underbrace{\text{b-spheres}}_{\text{double of ribbon 2-disks}}$ of 2-handles of D^5.

$\Rightarrow$ the Gluck twist preserves ∂D^5, i.e. $\underset{\Sigma(f)}{\overset{\parallel}{\partial P}} \cong \overset{S^4}{\overset{\parallel}{\partial D^5}}$ ///

Application :

Corollary : Let $h : H \xrightarrow[\partial_0]{\cong} H = H_n$ $(n = 2k)$, fixing $\partial_0 H$, $\phi = f_*$ is geom.

Then $D(\Sigma(h)) = S^4$

Consequently, the $\mathbb{Z}HS^3$ $\partial\Sigma(h)$ embeds in S^4.

If, in addition, $\partial\Sigma(h) = S^3$, then $\Sigma(h)$ is a Schoenflies ball i.e. a homotopy 4-ball that embeds in S^4.

- Is the above $\Sigma(h)$ diffeo. to D^4 whenever $\partial\Sigma(h) = S^3$? (There is a large family of h s.t. $\partial\Sigma(h) = S^3$!)

- Freedman's strategy to (disprove) SPC4:

  A $\mathbb{Z}HS^3$ bounds a contractible X^4 but doesn't embed in S^4 $\Rightarrow$ $D(X^4) \simeq S^4$ but $\mathbb{Z}HS^3 \subset D(X^4) \ncong S^4$

Further Question: Calegari spheres $\Sigma(f)$ from non-geometric automorphisms :

are these diffeo. to S^4 ?

$(\Sigma(f) \simeq S^4$ if $(P_\phi) \cong \{1\}$ for $\phi = f_*)$