Every negative amphichiral knot is rationally slice

joint work w/ J. Lee & O. Şavk

Def: $L_0, L_1 \subset S^3$ n-comp links rationally concordant if $\exists$

$C: A_1 \sqcup \dots \sqcup A_n \hookrightarrow S^3 \times I$ $\quad \mathbb{Q}H(S^3 \times I)$ $\quad C \cap S^3 \times \{0\} = L_0$

annuli $\quad C \cap S^3 \times \{1\} = L_1$

L_0 is $\mathbb{Q}$-slice if $L_0 \overset{\mathbb{Q}}{\sim} U^n$ n-comp. unlink

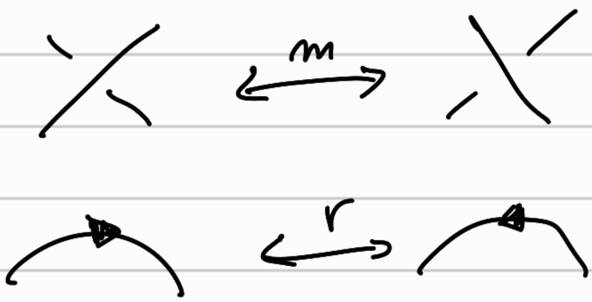
$\Longleftrightarrow$ L_0 bounds n disjoint disks in B^4 $\quad \mathbb{Q}HB^4$

Def: Concordance group $\quad \mathcal{C} = \left(\dfrac{\{\text{oriented knots}\}}{\text{concordance}}, \# \right)$

Rational conc. group $\quad \mathcal{C}_{\mathbb{Q}} = \mathcal{C} / \begin{smallmatrix}\mathbb{Q}\text{-slice} \\ \text{knots}\end{smallmatrix}$

Identity: $[U]$

Inverse: $-[K] = [mr(K)]$



Def: $L \subset S^3$ link $(K_1 - K_{p+q})$ (p,q)-amphichiral link

$\exists f: S^3 \to S^3$ or. rev. s.t. $f(K_i) = -K_i$ $i = 1 - p$

inverts the orients on S^1

$f(K_i) = +K_i$ $i = p+1 - p+q$

$q = 0 \longrightarrow L$ is neg. amph.

$p = 0 \longrightarrow L$ is pos. amph.

If f can be taken so that $f^2 = id_{S^3}$ $\to L$ is strongly (p,q)-amph.

$\hookrightarrow$ if this is not the case $\to$ weakly

Remark: K neg. amph knot $\Rightarrow K = mr(K)$

$[K] = -[K] \in \mathcal{C} \implies 2[K] = 0 \in \mathcal{C}$

$\rightsquigarrow \mathbb{Z}/_2^{\infty} \subset \mathcal{C}$

Conj (Gordon '76): $\mathrm{Tors}(\mathcal{C}) = \langle$ neg. amph. knots $\rangle$

Thm (Cochran, Fintushel-Stern): $\underset{\substack{\neq \\ 0}}{[4_1]} \in \mathrm{Ker}(\mathcal{C} \to \mathcal{C}_{\mathbb{Q}})$

$\downarrow$

Thm (Cha '07): $\mathbb{Z}/_2^{\infty} \subset \mathrm{Ker}(\mathcal{C} \to \mathcal{C}_{\mathbb{Q}})$

Thm (Kawauchi '09): K strongly neg. amph. knot $\Rightarrow$ $\mathbb{Q}$-slice

V Kawauchi mfd $\quad V = \mathbb{Z}[\frac{1}{2}] H B^4$

Thm (HKPS '22): $\mathbb{Z}^\infty \subset \ker(\mathcal{C} \to \mathcal{C}_\mathbb{Q})$

Q: Are the 2-torsion elements in $\mathcal{C}_\mathbb{Q}$?

$\rightsquigarrow$ weakly neg. amph. knots

Thm (Kawauchi '86): K neg. amph. & hyp $\Rightarrow$ strongly neg. amph.

[Hartley '87]: provided the first examples of weakly neg. amph. knots

Thm (Kim, Wu '16): K $\underbrace{\text{fibered, neg. amph. } \Delta_K(t) \text{ irred}}$ $\rightarrow$ $\mathbb{Q}$-slice

$\hookrightarrow$ Miyazaki knots

Q1: Is every neg. amph knot $\mathbb{Q}$-slice?

Q2: Is every Miyazaki knot strongly neg. amph.?

Thm A: Every neg. amph. link L is $\mathbb{Q}$-slice

(is slice in $\natural^n V$)

fibered, hyp

Thm B: K neg. amph knot w/ $\overbrace{\text{"totally coherent JSJ str"}}$

$\Rightarrow$ K is strongly neg. amph.

Ingredients

- JSS J decomposition: M^3 cpt. orient. irred. with torus ∂

$\longrightarrow \exists!$ up to isotopy $\tau = \{T_i\}$ minimal set of incompressible tori

s.t. τ splits M into Seifert fibered or hyperbolic pieces

- Gromov norm: $\|M\| = \sum_{\substack{Y \in JSJ(M) \\ Y \text{ hyp}}} Vol_{hyp}(Y)$

$\hookrightarrow$ The set of $\{\|M\| \mid M^3\}$ is a well ordered subset of $\mathbb{R}^+$

Proof of Thm A: $L \subset S^3 \circlearrowleft f$ neg. ampl. $\longrightarrow L$ is non-split

$$E_L = S^3 - \nu(L) \text{ is irred.}$$

0. isotope f s.t. $f(\tau) = \tau \longrightarrow f$ preserves the JSJ dec. of E_L

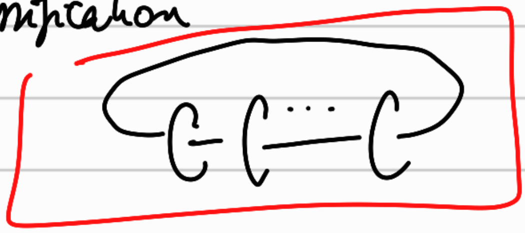
1. Suppose $\exists!$ piece Y of $JSJ(E_L)$ $f(Y) = Y$

$\hookrightarrow$ totally coherent JSJ str.

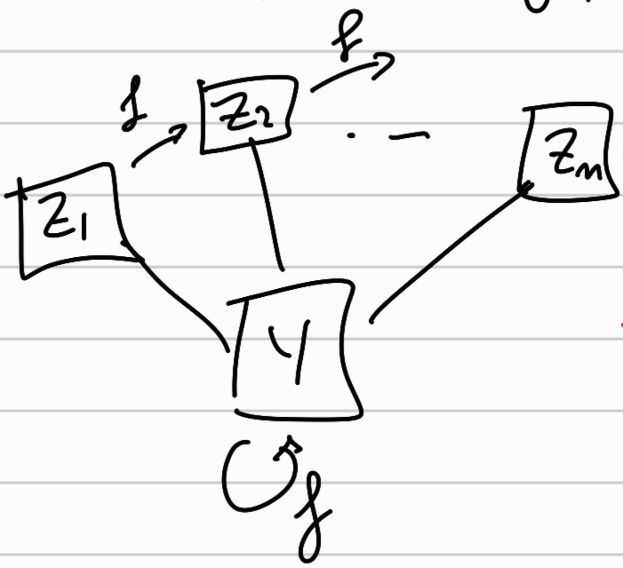
$\hookrightarrow$ modify f so that $f^2 = id$

Idea: • Y Seifert fibered $\longrightarrow$ classification

↳ modify $f|_Y$ so that $f|_Y^2 = \mathrm{id}$



• Y hyp. $\longrightarrow$ $f|_Y$ is an isometry $\longrightarrow$ finite order ↳ 2^k



$$\boxed{|\mathrm{Orb}(Z_1)| = 2^k}$$

$$Z_1 \xrightarrow{f} f(Z_1) \xrightarrow{f} f^2(Z_1) \to \dots \xrightarrow{f} f^{2^k-1}(Z_1), \qquad f^{2^k-1}(Z_1) \xrightarrow{f^{1-2^k}} Z_1$$

new map has order 2^k

⤳ we get a map f of order 2^k on E_L

→ Smith theory of fixed points $\longrightarrow$ $k=1$

Remark: There is always at least one Y s.t. $f(Y) = Y$

for instance $Y \cap \partial E_L \neq \emptyset$

2. Proceed by induction on $(\|E_L\|, \# JSJ(E_L))$

If 1. does not hold

$\rightsquigarrow L = L_1 \bowtie L_2$

$L_1 = K_0 \cup \dots \cup K_p$ $\quad$ $L_2 = J_0 \cup \dots \cup J_q$

$$(S^3, \underbrace{L}) = (S^3 - \nu(K_0), \underbrace{K_1 - K_p}) \cup_\partial (S^3 - \nu(J_0), \underbrace{J_1 - J_q})$$

$m_{K_0} \sim \lambda_{J_0}$

$\lambda_{K_0} \sim m_{J_0}$

$\rightsquigarrow$ one of the links is $(p/q, 1)$ - amph

$(q/p, 0)$ - amph

Assume that $f(K_0) = -K_0$ $\quad$ $f(J_0) = +J_0$

$$(\|E_{L_{1,2}}\|, \# JSJ(E_{L_{1,2}})) < (\|E_L\|, \# JSJ(E_L))$$

$\longrightarrow$ by induction $L_1 \overset{\mathbb{Q}}{\sim} U^{p+1}$

$\Longrightarrow L = L_1 \bowtie L_2 \overset{\mathbb{Q}}{\sim} U^{p+1} \bowtie L_2 = U^p \amalg (L_2 - J_0)$

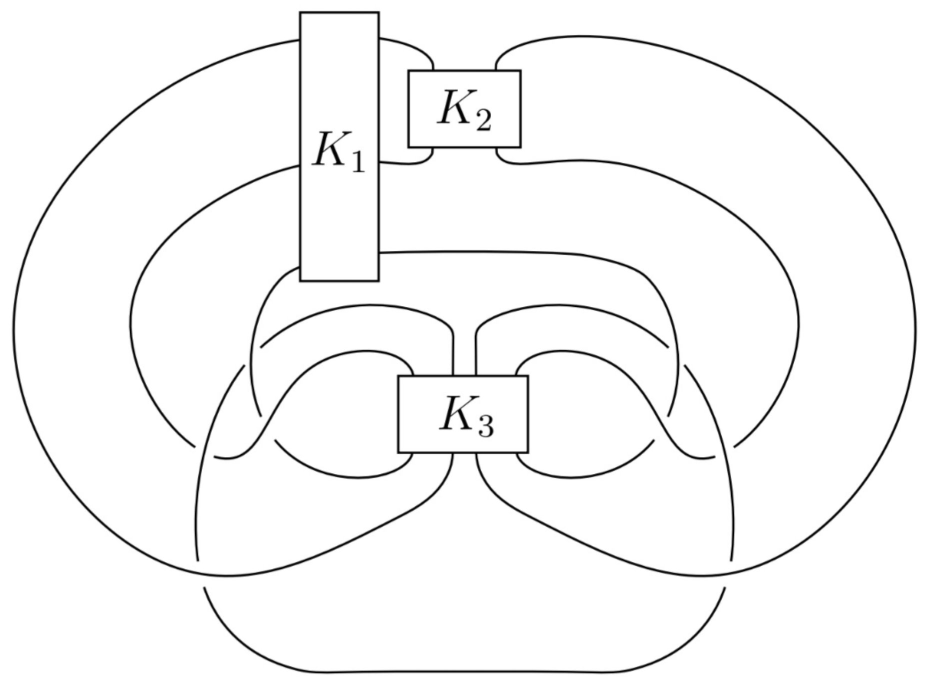
neg. amph. link

$\rightsquigarrow$ induction $L_2 - J_0 \overset{\mathbb{Q}}{\sim} U^q \Longrightarrow L$ is $\mathbb{Q}$-slice

Q1: Is every neg. amph knot concordant to a strongly neg. amph. knot?

Q2: Does there exist a neg. amph. knot K which is not slice in V? $\not\exists^{n} V$

Yes to Q2 $\Longrightarrow$ No to Q1



$= J(K_1, K_2, K_3)$

K_1 neg. amph.

K_2, K_3 pos. amph

$\rightarrow$ candidate to Q1

$\rightarrow \exists\, K_1', K_2', K_3' \rightsquigarrow J(K_1,K_2,K_3) \# J(K_1',K_2',K_3')$

candidate to Q2

$\partial V = S^3 \rightsquigarrow$ fill $Z = S^2 \times S^2 / a \times r$

unique S^2-bundle over $\mathbb{RP}^2$ ...