

Non-smoothable surfaces in S^4

Conv: Everything is topological.

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k knot $S^1 \hookrightarrow S^3$ is unknotted if it bounds a disc.



$$\pi_1(S^3 \setminus k) \cong \mathbb{Z}$$

Question: Does $\pi_1(S^3 \setminus k) \cong \mathbb{Z}$
 $\Downarrow$?
 $k = u$.

Ans: Yes, by Dehn's lemma.

Analogue in dim 4

$$F \hookrightarrow S^4$$

• If F is orientable ($F = \Sigma_g$) then F is unknotted if it bounds a handle body.



• We define $U_{\pm}$ the unknotted $\mathbb{RP}^2 \hookrightarrow S^4$ as:

$$\mathbb{CP}^2 \rightarrow \frac{\mathbb{CP}^2}{\mathbb{Z}_2} \cong S^4$$

branched double cover
 $U_{\pm} = \text{branch locus.}$

We say F non-orientable is unknotted if $F = \#U_+ \#U_-$.

• $\pi_1(S^4 \setminus F) \cong \mathbb{Z}$ if F orientable
 $\uparrow$
unknotted
 $\mathbb{Z}_2$ if F non-orientable

Question: Does $\pi_1(S^4 \setminus F)$ cyclic
(+) $\Downarrow$?
 F is unknotted.

Ans No.

Theorem [Conway-G. '76]

$\exists F: \#_q \mathbb{RP}^2 \hookrightarrow S^4$ such that

$$\pi_1(S^4 \setminus F) \cong \mathbb{Z}_2.$$

• F is not isotopic to any smooth embedding.

Corollary A $\exists$ non-smoothable surface in S^4 .

Corollary B

(+) is false for all non-orientable surfaces w/ genus $h \geq 9$.

Remarks

• (H) was known to be true

$$L F = S^2 \text{ (Freedman)}$$

$$L F = \Sigma_g \quad g \geq 3 \text{ (Conway-Powell)}$$

$$L F \text{ non-orientable genus } h \leq 5$$

(Conway-Orson-Powell + Pencovitch)
Lawson ($h=1$)

• known since 80s that there exist non-smoothable surfaces in other 4-manifolds e.s. $S^2 \times S^2$
(Rudolph, Kuran, Luo, LW...)

Overview

$\left\{ \begin{array}{l} \text{surfaces} \\ \text{in } S^4 \end{array} \right\}$ $\xrightarrow{\text{double branched cover}}$ $\left\{ \begin{array}{l} 1\text{-connected} \\ \text{closed 4-manifolds} \end{array} \right\}$
 $\xrightarrow{\text{isotopy}}$ $\xrightarrow{\text{homeo.}}$

$$F \longmapsto \Sigma_2(F).$$

Aim: Build an F such that

$$F \longmapsto \Sigma_2(F) \cong E_8 \# (\mathbb{CP}^2)^*$$

$$Q_{\Sigma_2(F)} \cong E_8 \oplus \langle 17 \rangle \cong \begin{pmatrix} 2 & & & & 0 \\ -12 & 2 & & & \\ & -12 & 2 & & \\ & & \ddots & \ddots & \\ 0 & & & 2 & 1 \end{pmatrix}$$

Donaldson

$\Rightarrow \Sigma_2(F)$ is not smoothable.

Preliminaries

$F \hookrightarrow S^4$ a π_2 -surface

Topology	Algebra
$\Sigma_2(F)$	$Q_{\Sigma_2(F)}$
$X_F = S^4 \setminus \nu F$	λ_{X_F} equivariant intersection on $\pi_2(X_F)$

$$H_2(\Sigma_2(F)) \cong \mathbb{Z}^h.$$

$$\pi_2 X_F \cong H_2(X_F; \mathbb{Z}[\pi_1/2]) \cong \mathbb{Z} \oplus \mathbb{Z}[\pi_1/2]^{h-1}$$

$\left\{ \begin{array}{l} \text{definite, symmetric} \\ \text{bilinear, non-degenerate} \\ \mathbb{Z} \text{-forms} \end{array} \right\} \xleftarrow{n \equiv 4 \text{ mod } 8 + "1"} \left\{ \begin{array}{l} \text{hermitian} \\ \mathbb{Z}[\pi_1/2] \text{-forms} \end{array} \right\}$
 $\xrightarrow{\text{pullback}}$

$$b = E_8 \oplus \langle 17 \rangle \longmapsto \lambda_b.$$

Lemma 1 If $F \hookrightarrow S^4$ such that

$$\lambda_{X_F} \cong \lambda_b, \text{ then}$$

$$Q_{\Sigma_2(F)} \cong b.$$

Lemma 2 If $F \hookrightarrow S^4$ such that

$$\lambda_{X_F} \cong_{\text{st.}} \lambda_b, \text{ then } \exists F' \hookrightarrow S^4$$

$$\text{such that } \lambda_{X_{F'}} \cong \lambda_b.$$

Proposition 3

Set $\frac{1}{2}u_+ =: u.$

$$\lambda_{X_u} \cong_{\text{st.}} \lambda_b.$$

(Can substitute $b = E_8 \oplus \langle 17 \rangle$

$\downarrow$
 $L \oplus \langle 17 \rangle$
 $\uparrow$ even, definite, symmetric ...)

Pf of lemma 2

$$\pi_2(X_F \# k(S^2 \times S^1)), \lambda_{X_F \# k(S^2 \times S^1)}$$

$$\begin{array}{c} \downarrow \cong \\ \gamma \left((\pi_2(X_F), \lambda_{X_F}) \oplus (\mathbb{Z}[\pi_2], \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}) \right)^k \\ \cong \downarrow \\ (\mathbb{Z} \oplus \mathbb{Z}[\pi_2]^{h-1}, \lambda_b) \oplus (\mathbb{Z}[\pi_2]^2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})^k \\ \quad \quad \quad \langle \alpha_i, \beta_i \rangle \end{array}$$

$$\text{Look at } \gamma^{-1}(\alpha_i) \cdot \gamma^{-1}(\beta_i) = \delta_{ii}$$

How we can use the sphere emb.
theorem of [Freedman-Quinn]
[Pauk-U. Nay - Teldner]

$\Rightarrow \exists$ embedded framed 2-spheres

A_i, B_i rep. $\gamma^{-1}(\alpha_i), \gamma^{-1}(\beta_i)$

$$\subset X_F \# k(S^2 \times S^1)$$

$$\text{AND } A_i \cdot B_j = \delta_{ij}$$

Cut out res. nbhd of

$$\cup A_i \cup B_i$$

$\leadsto V$.

$$\text{Form } W := V \cup \gamma F.$$

Claim: $W \cong S^4$.

$$\Rightarrow W \stackrel{f}{\cong} S^4 \text{ by [Freedman]}$$

$$\text{Set } F' := f(F).$$

$\square$

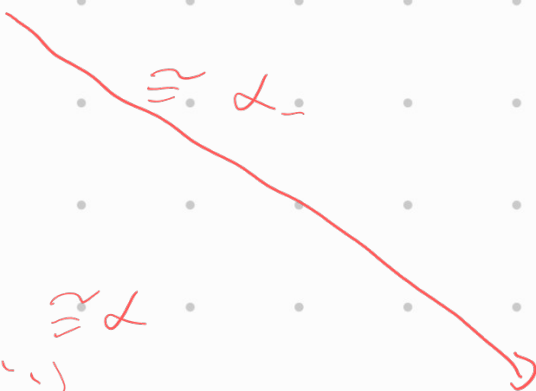
Pf of P3

Use pullback construction due to
Hambleton-Richm.

$$(\mathbb{Z} \oplus \mathbb{Z}[\pi_2]^{h-1}, \lambda_b) \longrightarrow (\mathbb{Z}_+^h, \lambda_{\text{even}})$$

$$\begin{array}{ccc} \downarrow & \lrcorner & \downarrow \\ (\mathbb{Z}_+^{h-1}, 0) & \longrightarrow & ((\mathbb{Z}_2)^{h-1}, 0) \end{array}$$

$$(\pi_2(x_u), \lambda_{x_u}) \oplus H \rightarrow (\sim) \oplus H$$



$$(\sim) \oplus H \rightarrow (\sim) \oplus H$$



$$((\sim), \lambda_b) \oplus H \rightarrow (\sim) \oplus H$$



$$(\sim) \oplus H \rightarrow (\sim) \oplus H$$

