Cobordism and Concordance of Surfaces in 4-Manifolds

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Key definitions

Let $\Sigma_0, \Sigma_1 \subset X^4$ be properly embedded compact surfaces. They are:

(i) *cobordant* if $\exists$ compact $Y^3 \subset X \times I$ with $Y \cap (X \times \{i\}) = \Sigma_i \times \{i\}$ for $i = 0, 1$;

(ii) *concordant* if $\Sigma_0 \cong \Sigma_1$ and $Y \cong \Sigma_0 \times I$.

isotopic $\Rightarrow$ concordant $\Rightarrow$ cobordant $\Rightarrow$ $\mathbb{Z}/2$-homologous

- ⚠ If Σ_0, Σ_1 oriented and Y can be taken to be oriented, say they are *oriented cobordant*.
- ⚠ No difference between topological and smooth categories. [Freedman–Quinn '90, Daher–Powell '23]

Talk structure

(i) Classification up to cobordism, X orientable.

Find set of obstructions.

Build cobordisms out of spanning 3-manifolds for $\Sigma_0 \cup \Sigma_1$ away from intersections.

Reduce to a combinatorial problem!

(ii) Classification up to concordance, $\pi_1(X) = 1$.

Use integral Dehn surgery to convert a cobordism Y into $\Sigma \times I$.

Show these surgeries can be realised ambiently in $X \times I$.

Conventions:

- X is oriented.
- Everything is smooth.
- For notational ease, $\partial X = \varnothing$ (NB: boundary versions exist!).

Oriented cobordism

Suppose Σ_0 and Σ_1 are oriented.

Theorem [Thom '54]

$\Sigma_0, \Sigma_1 \subset X$ are oriented cobordant iff they are $\mathbb{Z}$-homologous.

Proof sketch. Some transversality; then just homotopy theory.
Key point: $K(\mathbb{Z}, 2) \simeq \mathrm{BSO}(2) \quad (\simeq \mathbb{CP}^\infty)$.

What goes wrong for unoriented cobordism?

- In the proof: nothing to play the role of $\mathbb{CP}^\infty$.
- Obvious obstruction: unoriented cobordism group $\Omega_2^O \cong \mathbb{Z}/2$, e.g. $\mathbb{RP}^2$ and S^2 aren't even abstractly cobordant.

  So definitely can't have cobordant $\iff$ $\mathbb{Z}/2$-homologous.
- Less obvious obstruction: two embeddings $\mathbb{RP}^2 \hookrightarrow S^4$ which differ by a reflection aren't cobordant!

What goes wrong for unoriented cobordism?

Definition: normal Euler number

Let $\Sigma_i' \subset X$ be a small perturbation of Σ_i such that $\Sigma_i \pitchfork \Sigma_i'$. The *normal Euler number* $e(\Sigma_i) \in \mathbb{Z}$ is the number of intersections, counted with sign ($T_p\Sigma_i \oplus T_p\Sigma_i' = \pm T_pX$).

- E.g. $e(\mathbb{CP}^1 \subset \mathbb{CP}^2) = +1$; $e(\mathbb{RP}^2 \subset S^4) = \pm 2$.
- If $\Sigma \subset X$ orientable, then $e(\Sigma) = [\Sigma] \cdot [\Sigma]$, where $[\Sigma] \in H_2(X; \mathbb{Z})$.

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Proposition

If Σ_0, Σ_1 are cobordant, then $e(\Sigma_0) = e(\Sigma_1)$.

Proof sketch. Take a cobordism $Y \subset X \times I$, and consider the intersections of $Y \pitchfork Y'$. These form circles, arcs from $\Sigma_i \times \{i\}$ to itself, and arcs from $\Sigma_0 \times \{0\}$ to $\Sigma_1 \times \{1\}$.

Cobordism classification

Theorem A [H. '26]

$\Sigma_0, \Sigma_1 \subset X$ are cobordant iff they are $\mathbb{Z}/2$-homologous and $e(\Sigma_0) = e(\Sigma_1)$.

- [Carter–Kamada–Saito–Satoh '02]: $\Sigma_0, \Sigma_1 \subset \mathbb{R}^4$ connected are cobordant iff $e(\Sigma_0) = e(\Sigma_1)$, via broken surface diagrams & Roseman moves.
- If $\partial X \neq \varnothing$, can drop the Euler number condition by varying the boundary of the cobordism in $\partial X \times I$.

Inspiration: cobounding implies cobordant

- Best case: Σ_0, Σ_1 disjoint. If $\exists\ Y^3 \subset X$ such that $\partial Y = \Sigma_0 \sqcup \Sigma_1$, then Σ_0 and Σ_1 are cobordant.
- Thom-style argument with $\mathbb{RP}^\infty$ instead of $\mathbb{CP}^\infty$: this happens iff they are $\mathbb{Z}/2$-homologous and $e(C) = 0$ for each component $C \subseteq \Sigma_0 \sqcup \Sigma_1$.

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What about intersections?

- Locally, look like $(D^2 \times 0) \pitchfork (0 \times D^2) \subset D^4$. Meet $S^3 = \partial D^4$ in a Hopf link.
- Rough idea: resolve intersections by replacing pair of intersecting discs with an annulus. Connects different components $\Rightarrow$ "spreads out" Euler number condition.
- ⚠ Need to make sure spanning 3-manifold stays away from intersections, i.e. "outside" of annulus.

Main idea 1: almost-cobounding implies cobordant

Definition: almost-cobounding surfaces

Let $\widehat{\Sigma}_0, \widehat{\Sigma}_1 \subset \widehat{X}$ be the disjoint surfaces given by puncturing X around intersections.
We say Σ_0, Σ_1 are *almost-cobounding* if $\exists\ \widehat{Y}^3 \subset \widehat{X}$ such that $\partial \widehat{Y} = \widehat{\Sigma}_0 \cup \widehat{\Sigma}_1 \cup \{\text{annuli}\}$.

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- Σ_0, Σ_1 are almost-cobounding $\Rightarrow \widehat{\Sigma}_0, \widehat{\Sigma}_1$ are cobordant.
- We can fill in the punctures: $\Rightarrow \Sigma_0, \Sigma_1$ are cobordant.

  $\Rightarrow \Sigma_0, \Sigma_1$ are $\mathbb{Z}/2$-homologous, $e(\Sigma_0) = e(\Sigma_1)$.

Want to show that $\mathbb{Z}/2$-homologous & $e(\Sigma_0) = e(\Sigma_1) \Rightarrow$ almost-cobounding (up to isotopy).

Main idea 2: almost-cobounding is purely combinatorial!

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- Σ_0, Σ_1 are almost-cobounding iff $\widehat{\Sigma}_0 \cup \widehat{\Sigma}_1 \cup \{\text{annuli}\}$ is null-homologous and each component has zero normal Euler number.
- Normal Euler number depends only on choice of annuli – combinatorics!

  Two different choices of annulus change normal Euler number by ±2.

  Each component has zero normal Euler number iff there is a choice of $\varepsilon_i \in \{\pm 1\}$ for each intersection $p_i \in \Sigma_0 \pitchfork \Sigma_1$ such that for each component C of Σ_0 or Σ_1,

$$e(C) = \sum_{p_i \in C} \varepsilon_i.$$

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- Performing finger moves between Σ_0 and Σ_1 changes them by an isotopy, but introduces more intersections $\Rightarrow$ more freedom.

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Proposition

Σ_0 and Σ_1 are almost-cobounding up to isotopy iff they are $\mathbb{Z}/2$-homologous and $e(\Sigma_0) = e(\Sigma_1)$, and hence iff they are cobordant.

Concordance classification: some history

Theorem [Kervaire '65]

Suppose $X = S^4$, $\Sigma_0 \cong \Sigma_1 \cong S^2$. Then $\Sigma_0, \Sigma_1 \subset X$ are concordant.

Concordance classification: some history

Theorem [Kervaire '65, Blanlœil–Saeki '05]

Suppose $X = S^4$, $\Sigma_0 \cong \Sigma_1$ connected. Then $\Sigma_0, \Sigma_1 \subset X$ are concordant iff $e(\Sigma_0) = e(\Sigma_1)$.

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Theorem [Sunukjian '15]

Suppose $\pi_1(X) = 1$, $\Sigma_0 \cong \Sigma_1$ connected & oriented. Then $\Sigma_0, \Sigma_1 \subset X$ are concordant iff they are $\mathbb{Z}$-homologous.

Concordance classification: the present day

Theorem B

Suppose $\pi_1(X) = 1$, $\Sigma_0 \cong \Sigma_1$ connected. Then $\Sigma_0, \Sigma_1 \subset X$ are concordant iff:

- [Sunukjian '15] they are orientable and $\mathbb{Z}$-homologous (up to sign); or
- [H. '26] they are non-orientable, $\mathbb{Z}/2$-homologous, and $e(\Sigma_0) = e(\Sigma_1)$.

That is, concordant $\iff$ (oriented) cobordant for simply-connected X.

If $\partial X \neq \varnothing$, $\Sigma_0 \cong \Sigma_1$ connected are concordant iff they are cobordant via a cobordism which is a concordance on the boundary.

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For $\pi_1(X) \neq 1$, see [Freedman–Quinn '90], [Stong '93], [Klug–Miller '21, '22].

Open problem

Are all 2-links $\bigsqcup^n S^2 \subset S^4$ concordant?

Outline of approach

- Assume (ori) cobordant, so $\exists$ (ori) $Y \subset X \times I$ such that $Y \cap X \times \{i\} = \Sigma_i \times \{i\}$.
- Relative Lickorish–Wallace: $\exists$ sequence of surgeries talking Y to $\Sigma \times I$. We want to realise these ambiently in $X \times I$.
- To realise surgery on $K \subset Y$ with framing φ ambiently, find a 2-disc $\Delta \subset X \times I$ with $\partial\Delta = \Delta \cap Y = K$, and a framing of a rank 2 subbundle of $N_{X \times I}\Delta$ extending φ.

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Two obstructions to realisability:

- Does there exist such a Δ for each K?
- Does there exist such a Δ for each framing φ?

Overcoming first obstruction

- Every K has a spanning 2-disc $\Delta \iff$ every K has a null-homotopic push-off into $(X \times I) \setminus Y$.

- If $\pi_1(X) = 1$, then $\pi_1(X \times I \setminus Y)$ is generated by meridians to Y.

  By performing ambient connected sums (0-surgeries), we can identify meridians at the endpoints of the connected sum arc.

⚠ Upshot: we can assume $\pi_1(X \times I \setminus Y)$ is cyclic and generated by a meridian to Y.

- Then every K has a null-homotopic push-off: choose any push-off, band sum to some number of meridians.

Overcoming second obstruction: part 1

Proposition

If there is a 2-sphere $S \subset (X \times I) \setminus Y$ with non-trivial normal bundle, then every 1-surgery can be performed ambiently in $X \times I$.

Proof sketch. For each (K, φ), find some Δ. If φ extends over Δ, we're done. If not, form $\Delta' = \Delta \# S$. Since $N_{X \times I} S$ is <u>the</u> non-trivial rank 3 bundle, framings of K extend over Δ' iff they don't extend over Δ.

Overcoming second obstruction part 2: Pin$^-$-structures

A Pin$^-$-structure on Y is $\approx$ unoriented Spin-structure:

- If Y is oriented, a Pin$^-$-structure and Spin-structure are the same data.
- Number of Pin$^-$-structures on $Y = |H^1(Y;\mathbb{Z}/2)|$.
- All compact 3-manifolds admit a Pin$^-$-structure.
- $\Omega_3^{\mathrm{Pin}^-} = \Omega_3^{\mathrm{Spin}} = 0$.

(Slightly) more precisely: a Pin$^-$-structure is a “compatible” choice of stable framing on $TY \oplus \det TY$ over every circle in Y.

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Fix a Pin$^-$-structure on Y. Call a surgery on (K, φ) a Pin$^-$-surgery if φ agrees with the specified stable framing (i.e. surgery preserves the Pin$^-$-structure).

Overcoming second obstruction part 2: Pin^--structures

Fix a Pin^--structure on Y. Call a surgery on (K, φ) a Pin^--surgery if φ agrees with the specified stable framing (i.e. surgery preserves the Pin^--structure).

Proposition

For any Pin^--structure on Y, there is a sequence of Pin^--surgeries taking Y to $\Sigma \times I$.

Proposition

If every 2-sphere in $(X \times I) \setminus Y$ has a trivial normal bundle, then $\exists$ a Pin^--structure on Y such that all Pin^--surgeries can be performed ambiently.

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If every 2-sphere in $(X \times I) \setminus Y$ has a trivial normal bundle, then $\exists$ a Pin^--structure on Y such that all Pin^--surgeries can be performed ambiently.

Corollary

There exists a sequence of surgeries converting any cobordism Y to a concordance.

Thank you for listening!

Any questions?