Exotic closed aspherical 4-manifolds

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joint with with M. Davis, K. Hayden, D. Ruberman and N. Sunukjian

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3. What if $n = 4$?

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3. We explain why the manifolds after closing up are not diffeomorphic.

Ingredient I: The reflection group trick of Davis

Let Γ be a finite simplicial graph with its vertex set $\{v_1, v_2, \ldots, v_n\}$. The associated *right-angled Coxeter groups* G_Γ is a group with generating set $\{v_i\}_{i=1}^n$ and the following two types of relators:

1. $v_i^2 = 1$ for $1 \leq i \leq n$;
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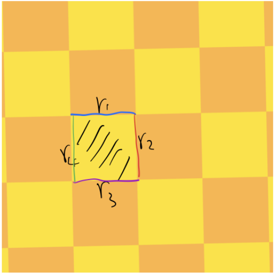
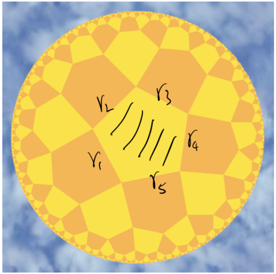
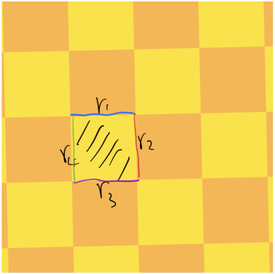
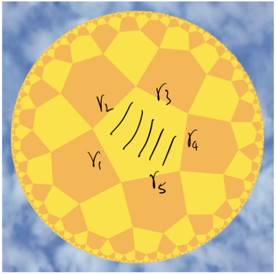




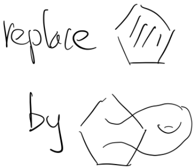
$W = \langle r_1, r_2, r_3, r_4 \mid r_i^2 = 1,\ r_i r_{i+1} = r_{i+1} r_i,\ 1 \leq i \leq 4 \}$



γ_1
γ_2
γ_3
γ_4



γ_1
γ_2
γ_3
γ_4
γ_5



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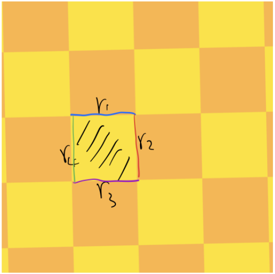
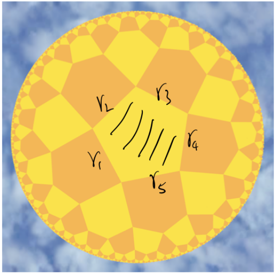
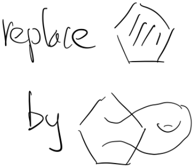
by







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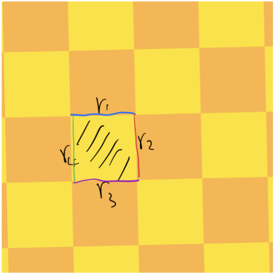

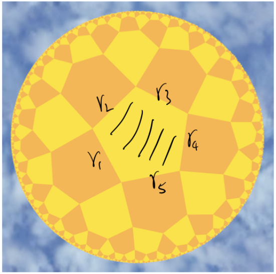

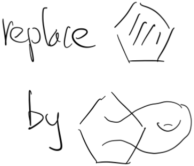

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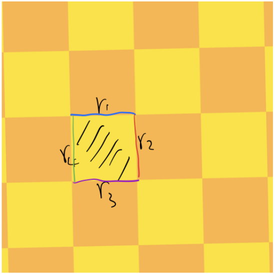

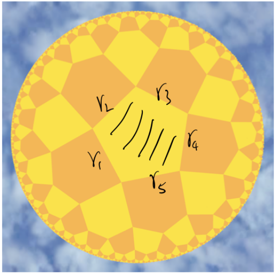

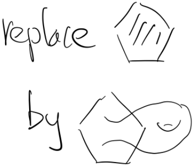

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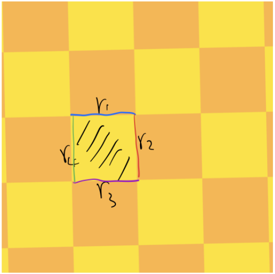

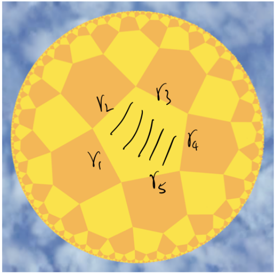

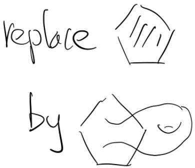

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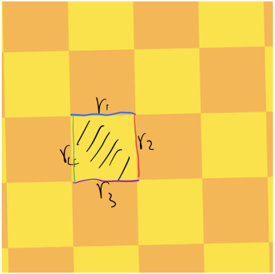

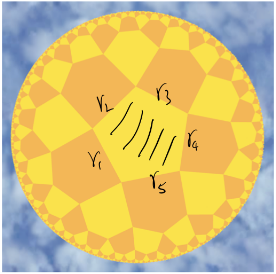

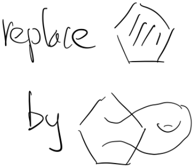

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Step 3: Take G' to be a finite index torsion free subgroup of G. Then $Q(X) = D(X)/G'$ is a closed manifold.

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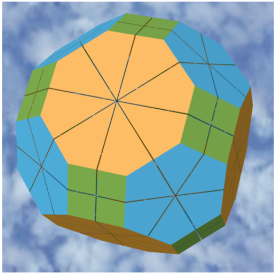

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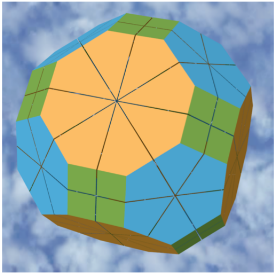

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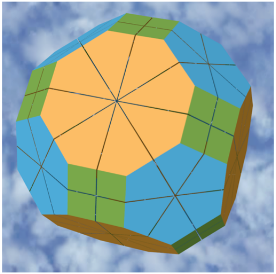

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The defining graph of the associated right-angled Coxeter group G is exactly the 1-skeleton of $\mathcal{T}$.

More about $D(X)$

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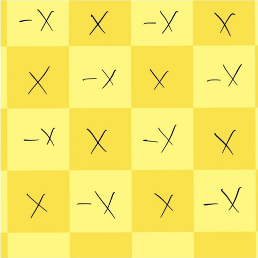

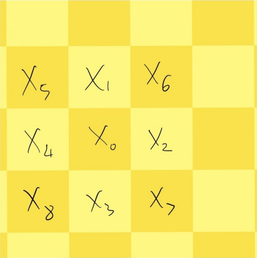

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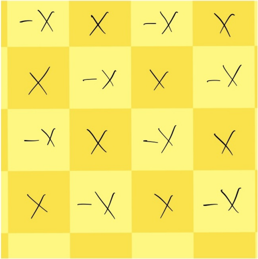

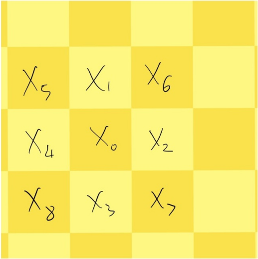

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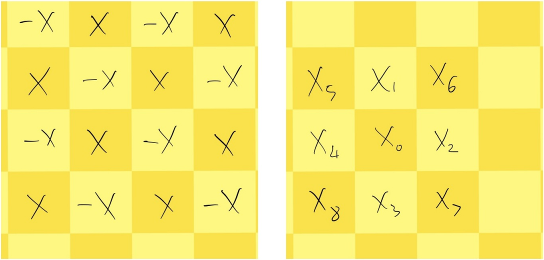

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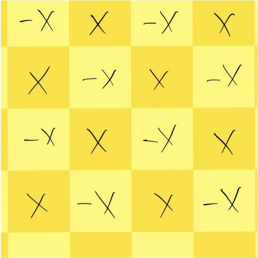
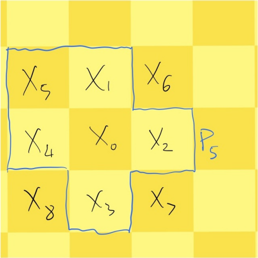
We enumerate copies of X in $D(X)$ as $X_0, X_1, X_2, \ldots$ such that $d(X_i, X_0) \leq d(X_{i+1}, X_0)$.

Properties of Davis construction





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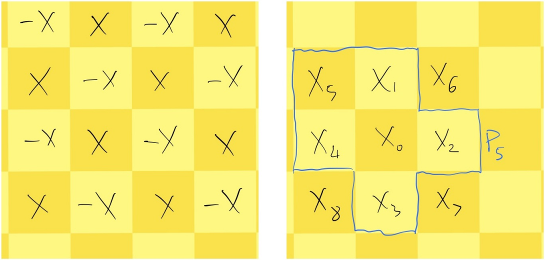

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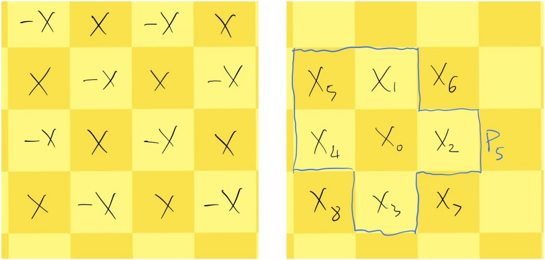

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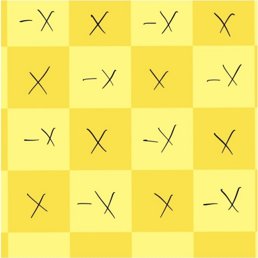

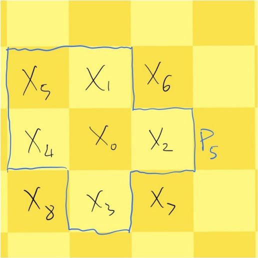

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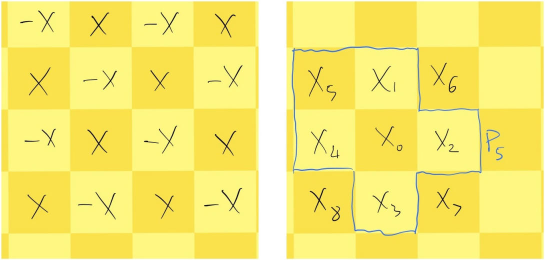

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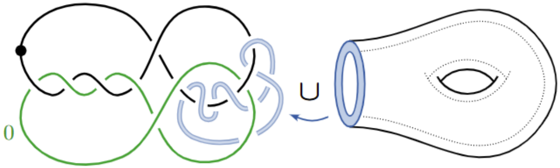
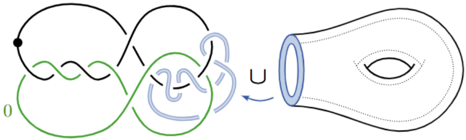
A genus 1-handle is a copy of $F \times \mathbb{D}^2$ where F is a genus-1 surface with one boundary component. We identify $\partial F \times \mathbb{D}^2$ with a tubular neighborhood of a knot K in ∂C.



X

Key properties of Hayden-Piccirillo manifolds



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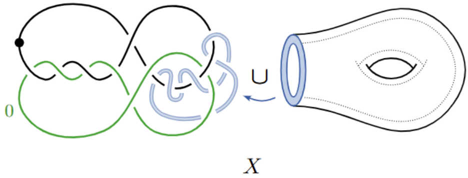

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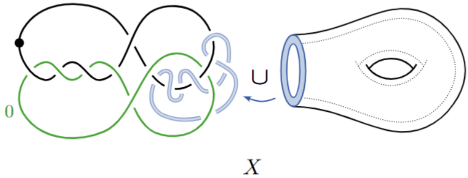

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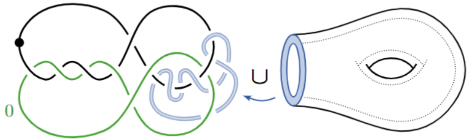

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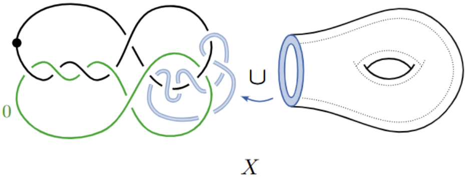

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4. *every homologically essential, smoothly embedded surface in X has genus ≥ 2;*
5. *$H_2(X')$ is generated by a smoothly embedded torus in X'.*

$D(X)$ vs $D(X')$

Observation: $D(X)$ and $D(X')$ are homeo, hence $Q(X)$ and $Q(X')$ are homeo.

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$T^2 \hookrightarrow f(X') \overset{PL}{\hookrightarrow} P_n \hookrightarrow X$, contradiction (using Wall's smoothing theorem for locally flat codimension-2 PL submanifolds of smooth manifolds).

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Lemma

For some particular choices of panel structures on $\partial X = \partial X'$, any $\mathbb{Z}^2$ subgroup of $\pi_1(Q(X))$ are contained in $\pi_1(D(X))$.
As a consequence, $\pi_1(D(X))$ is a characteristic subgroup of $\pi_1(Q(X))$.

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A *square* in $\mathcal{T}$ is an embedded 4-cycle $x_1x_2x_3x_4$ in $\mathcal{T}^{(1)}$ such that x_1 and x_3 are not joined by an edge adjacent, and so are x_2 and x_4.

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Theorem (Moussong 1988)

If G has a $\mathbb{Z}^2$-subgroup, then $\mathcal{T}$ has a square.

Flag no-square triangulation of manifolds

Cor: if ∂X admits a flag no-sqaure triangulation $\mathcal{T}$, then G' has no $\mathbb{Z}^2$-subgroup.

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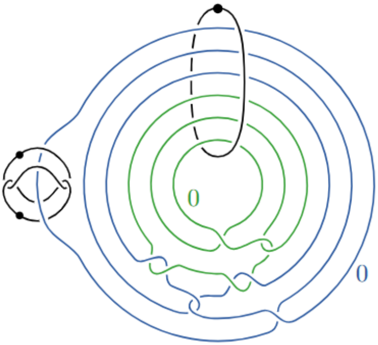
Theorem

Suppose X and X' are a pair of Hayden-Piccirillo manifolds (with genus 1 handles), with a flag no-square triangulation of $\partial X = \partial X'$. Then $Q(X)$ and $Q(X')$ are closed aspherical manifolds that are homeomorphic, but not diffeomorphic.

Thank you!

Kirby diagrams for X and X'



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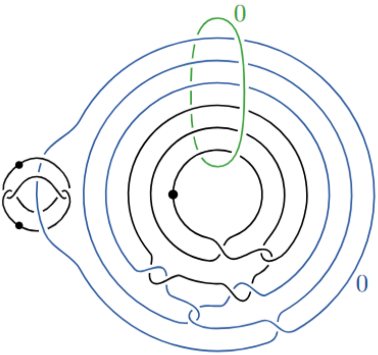

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