NONISOTOPIC SPLITTING SPHERES FOR SPLIT SURFACE LINKS

[K-OS] September 21, 2023

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joint with

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Outline:

1. Two uniqueness facts in S^3
2. Twist spun and doubly slice knots
3. Knotted handlebodies in S^4
4. Bing doubles and split links

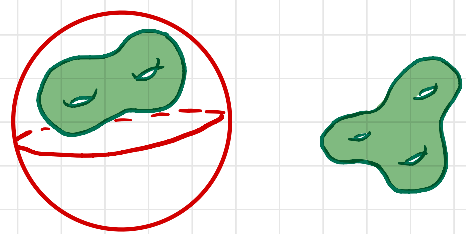
Two uniqueness facts in S^3:

Fact 1: The unknot $U \subseteq S^3$ bounds a unique disk in S^3 up to isotopy.

Def^n: A submanifold $L \subseteq S^n$ is *split* if $\exists$ an embedded $(n-1)$-sphere $\Sigma^{n-1} \subseteq S^n \setminus L$ such that both components of $S^n \setminus \Sigma$ intersect L.

In this case Σ is called a *splitting sphere* for L.

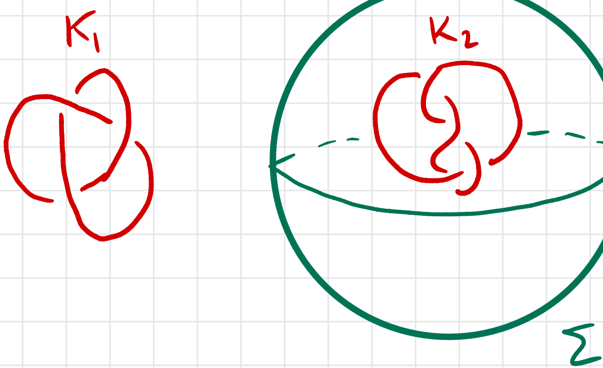
Σ



Fact 2: If $L = K_1 \sqcup K_2$ is a split 2-component link in S^3_1 then any splitting sphere for L is unique up to isotopy.

Proof: Inner-most disk argument.

$n=3$ Schoenflies $\Rightarrow$ any smoothly embedded $S^2 \subseteq S^3$ bounds a 3-ball.



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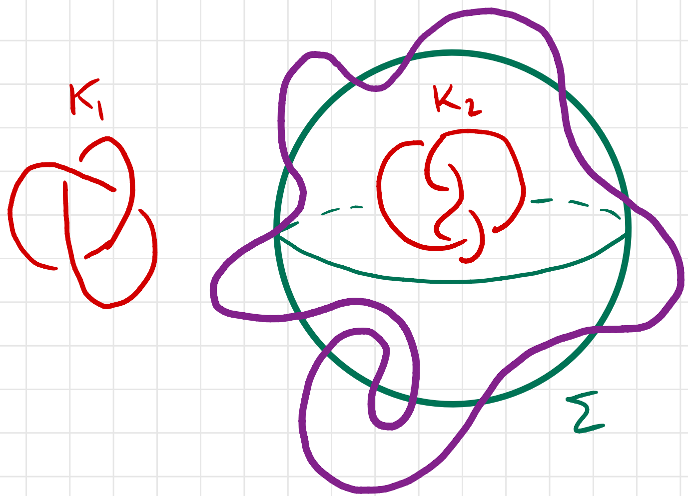

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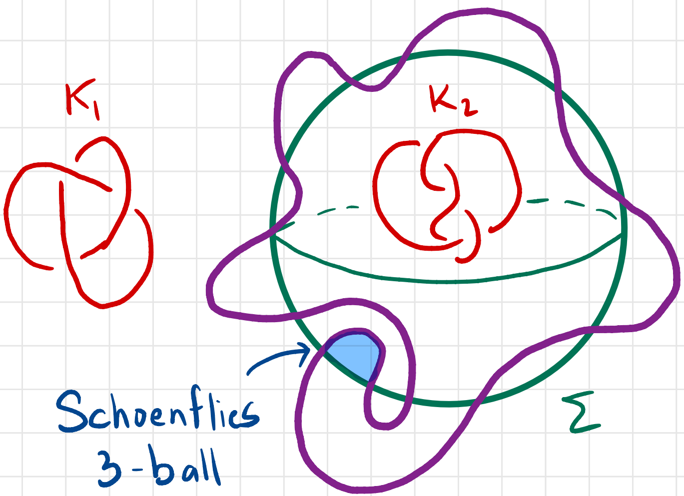

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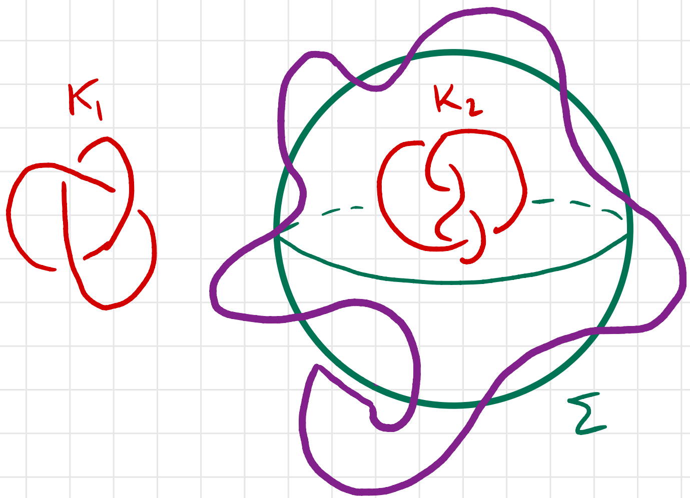

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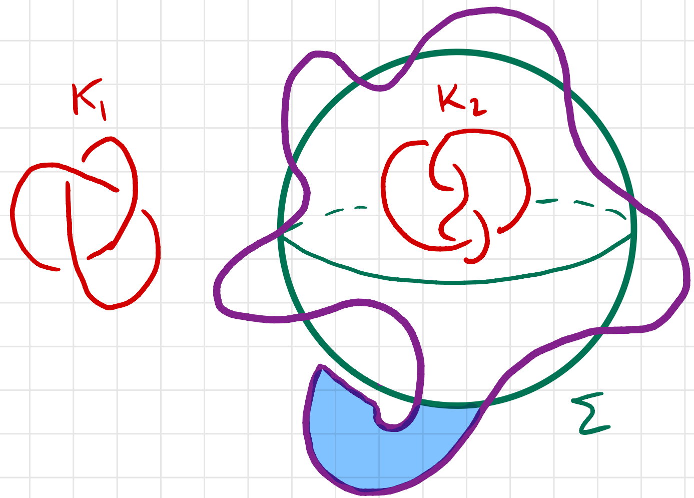

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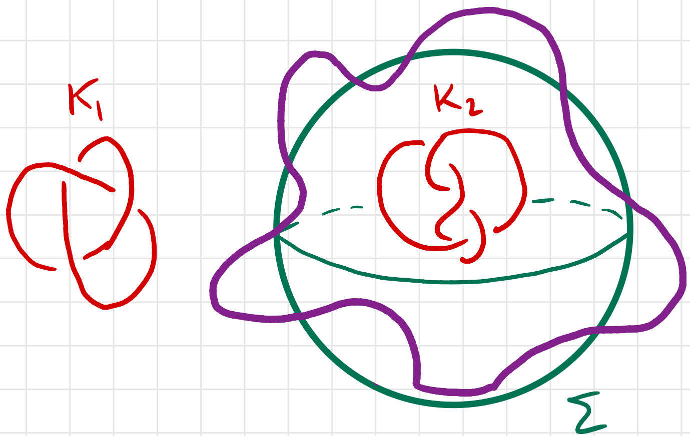

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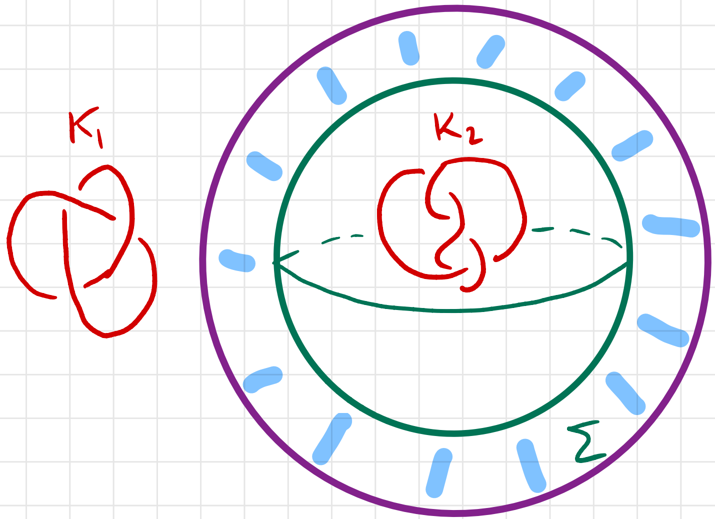

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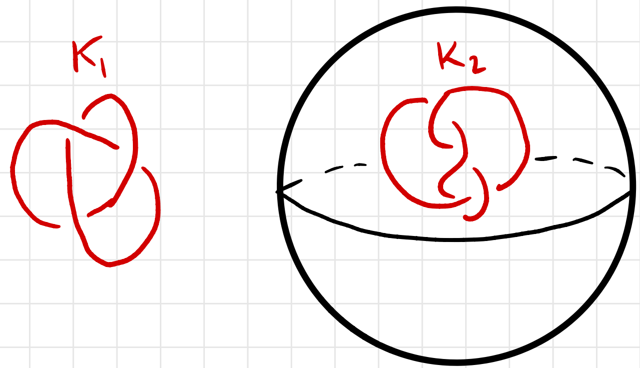
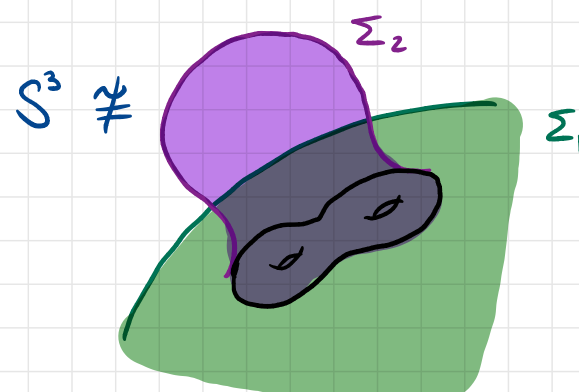


In ambient dimension $n \geq 4$ this argument fails.



For Σ_1, Σ_2 splitting spheres in S^4

$\Sigma_1 \cap \Sigma_2 =$ genus g surface

$\Longrightarrow$ can't use Schoenflies

In fact:

<u>Thm</u>: (Budney–Gabai '19) $\exists$ infinitely many 3-balls in S^4 with common boundary that are distinct up to smooth isotopy rel ∂.

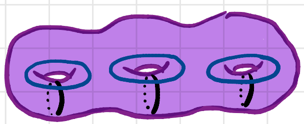
$\Rightarrow$ Thinking of $S^4 = \partial B^5$, the BG 3-balls become isotopic when pushed into B^5. In fact:

<u>Thm</u>: (Hartman '22) Any two 3-balls become isotopic rel ∂ when pushed into B^5.

Proof generalizes to n-balls in S^{n+1} for $n \geq 3$.

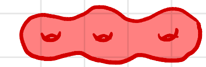
<u>Question</u>: What about higher genus surfaces in S^4?

Do they bound unique handlebodies up to isotopy rel boundary?



<u>Answer</u>: No!!! Different compressing curves $\Rightarrow$ Nonisotopic rel ∂.

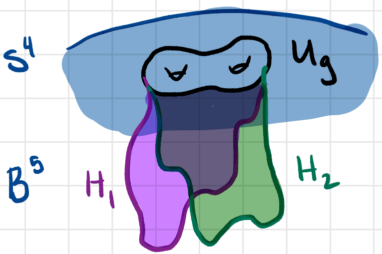
Defn: A surface link is a closed oriented surface L smoothly embedded in S^4.

L is unknotted if it bounds an embedded handlebody. 

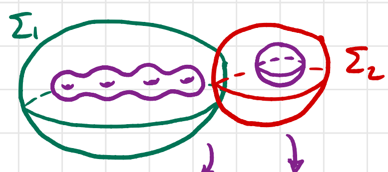
(Equivalently, if L can be isotoped to lie in the equatorial $S^3 \subseteq S^4$.)

$U_g :=$ genus g unknot

Thm: (H.-Kim-Miller '21) There exist handlebodies H_1 and H_2 in S^4 with boundary $\partial H_1 = \partial H_2 = U_g$ (for $g \geq 2$) that

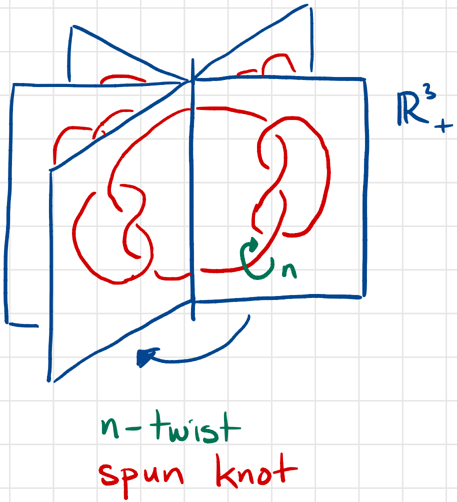


1. are compressing curve equivalent, and
2. are not isotopic rel ∂, even when pushed into B^5.

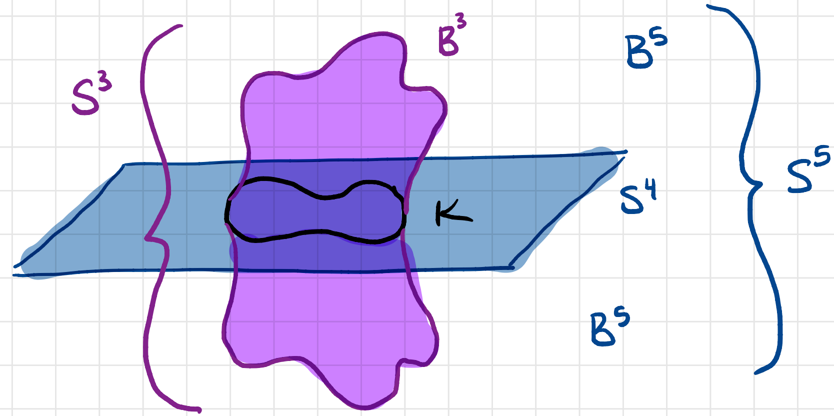


Thm: (H.-Kim-Miller '23) If $L = U_m \sqcup U_n$ with $m \geq 4$ then $\exists$ splitting spheres Σ_1 and Σ_2 for L which are not isotopic (smoothly or topologically).

Twist spun and doubly slice knots



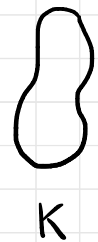
Thm: (Kervaire) Every smooth genus 0 surface knot $K \subseteq S^4 = \partial B^5$ bounds a smooth 3-ball B^3 in B^5



Def^n: (Doubly-slice) A genus 0 surface K knot is doubly-slice if $\exists$ an unknotted 3-sphere $P \cong S^3 \subseteq S^5$ such that $K = P \cap S^4$.

Thm: (Stolzfus, Ruberman) The 5-twist spun trefoil K_5 is not doubly slice.

Fact: (Satoh) K_5 can be converted to an unknotted torus by surgering on a single tube.



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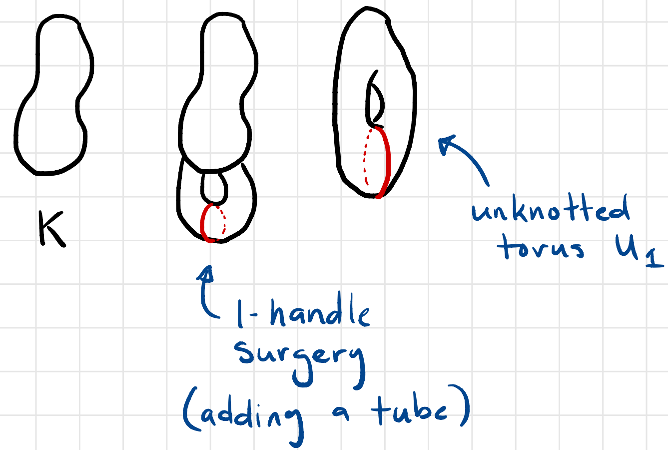

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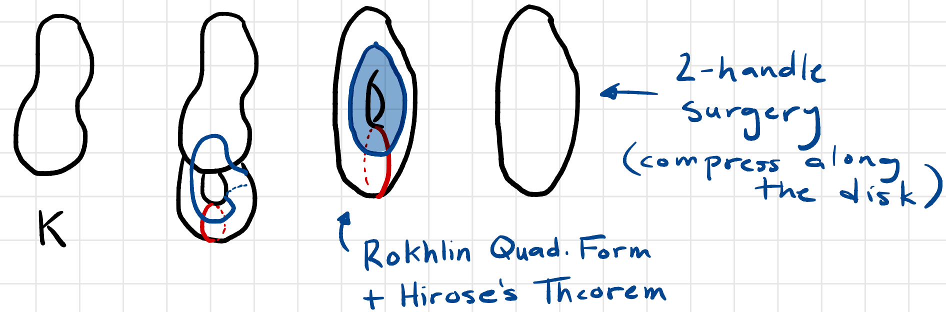

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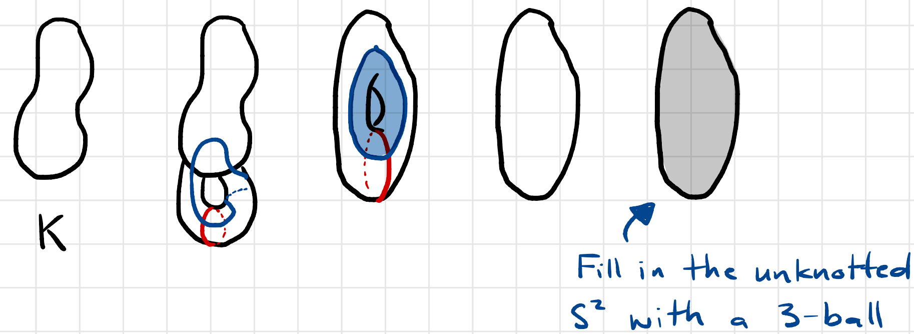

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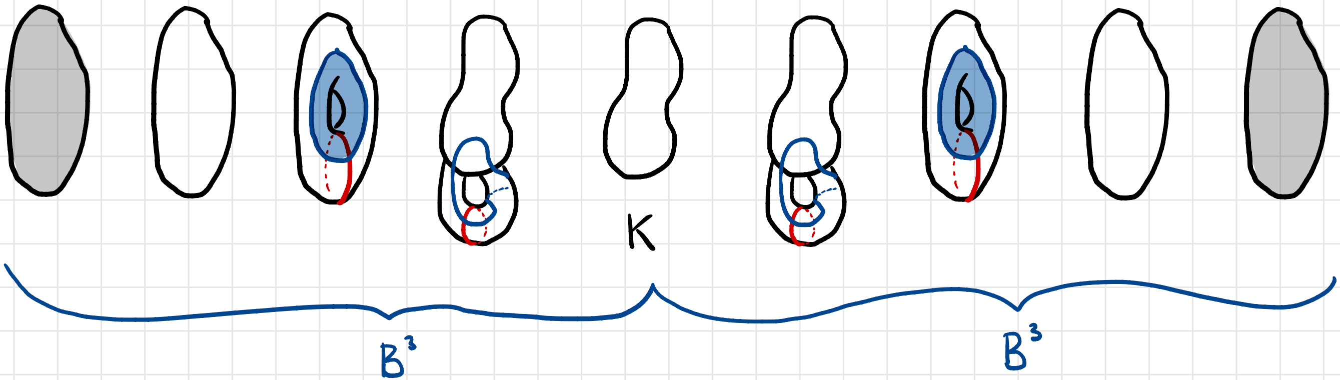
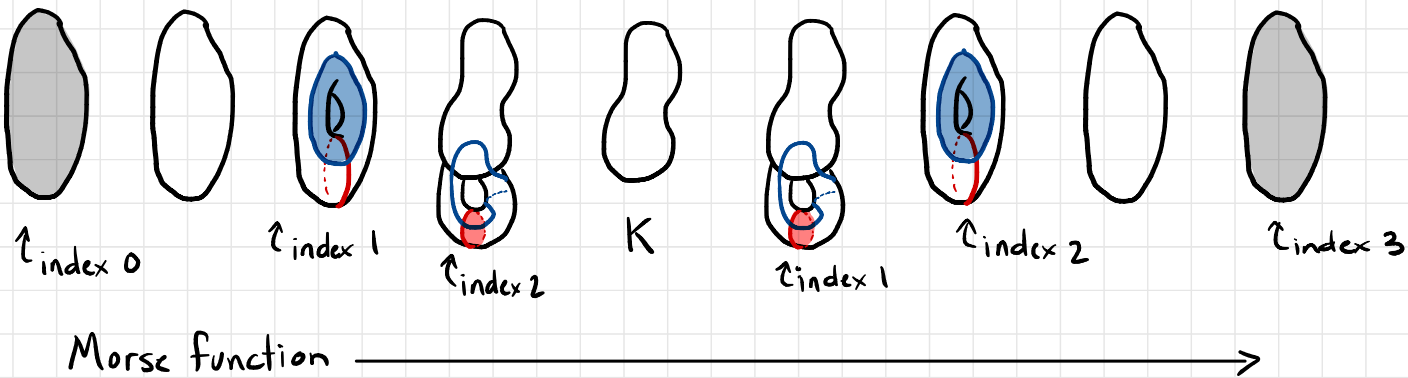


$$P = B^3 \cup B^3 \cong S^3 \text{ in } S^5$$

K_5 not doubly-slice $\Rightarrow$ P is knotted.

Fact: (Satoh) K_5 can be converted to an unknotted torus by surgering on a single tube.



$P \cong S^3$ knotted 3-sphere in S^5

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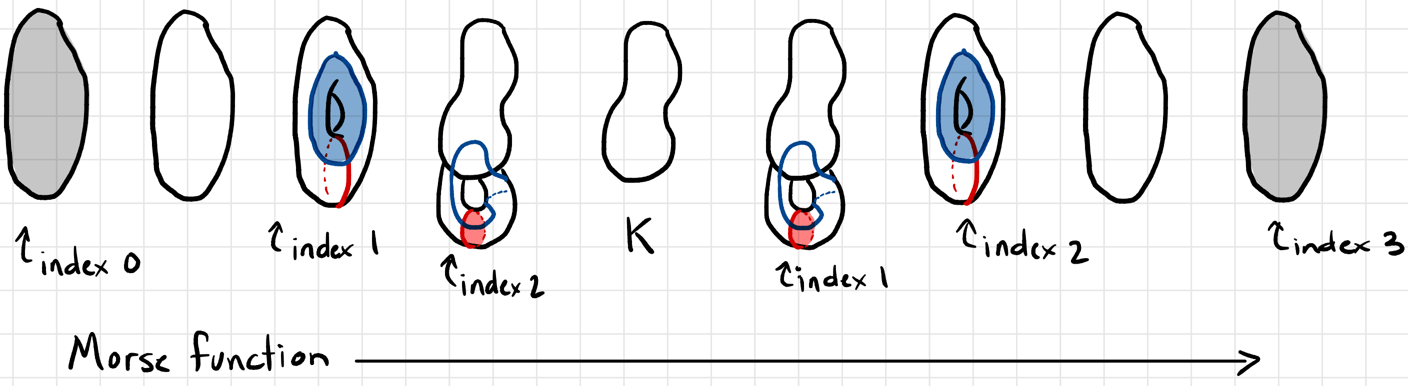

Reordering critical points: We can slide index 2 critical point down passed the index 1 critical point.

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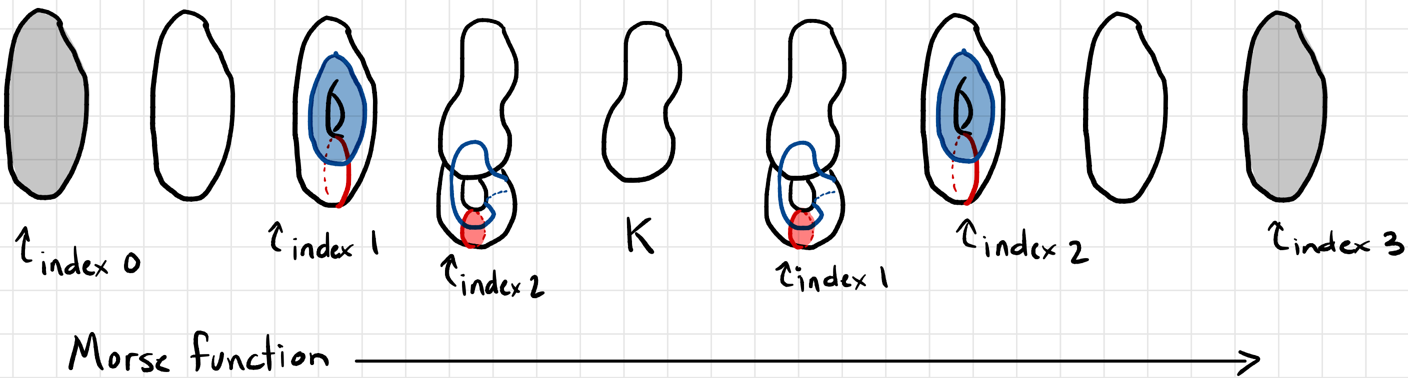

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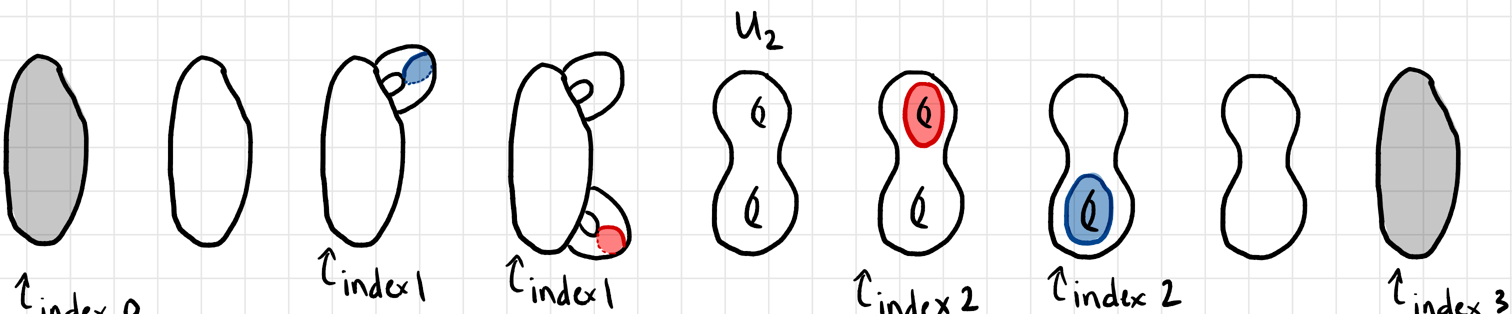
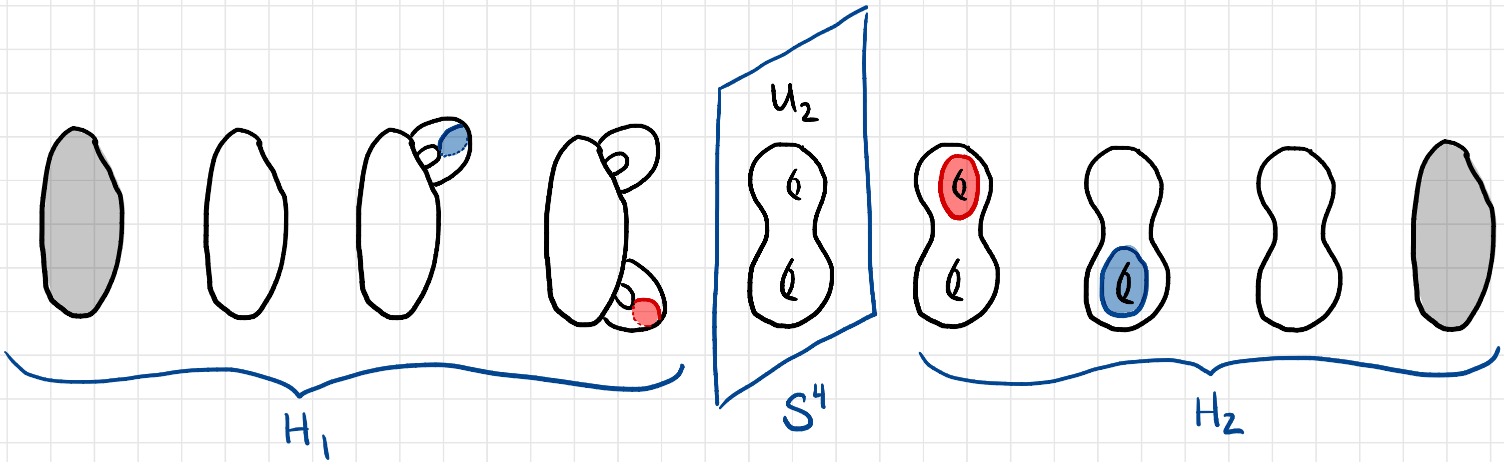
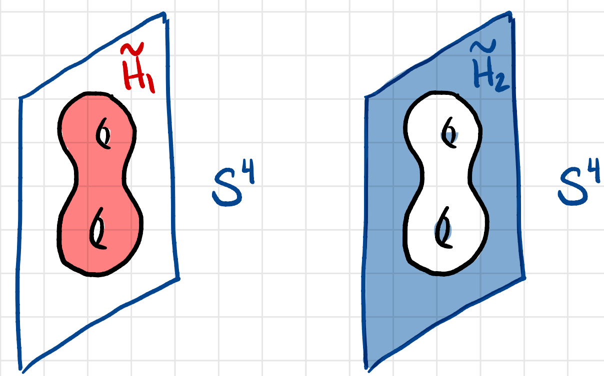
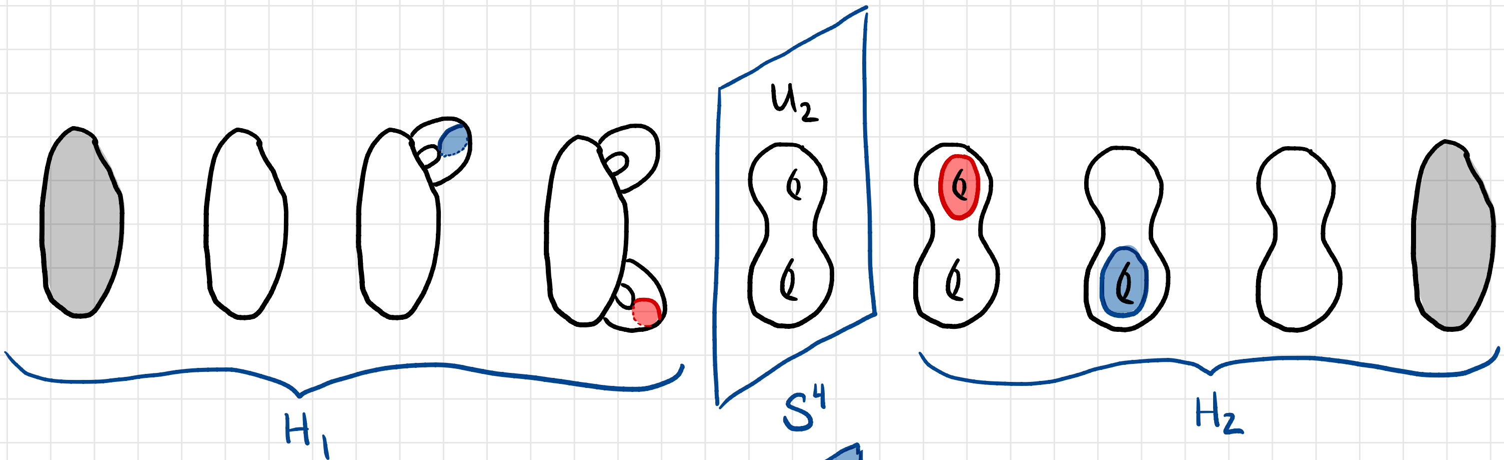




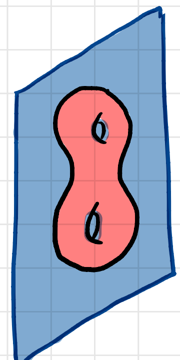
H_1 and H_2 are both handlebodies, both boundary parallel (i.e. H_1 and H_2 can be isotoped into S^4, but not disjointly).

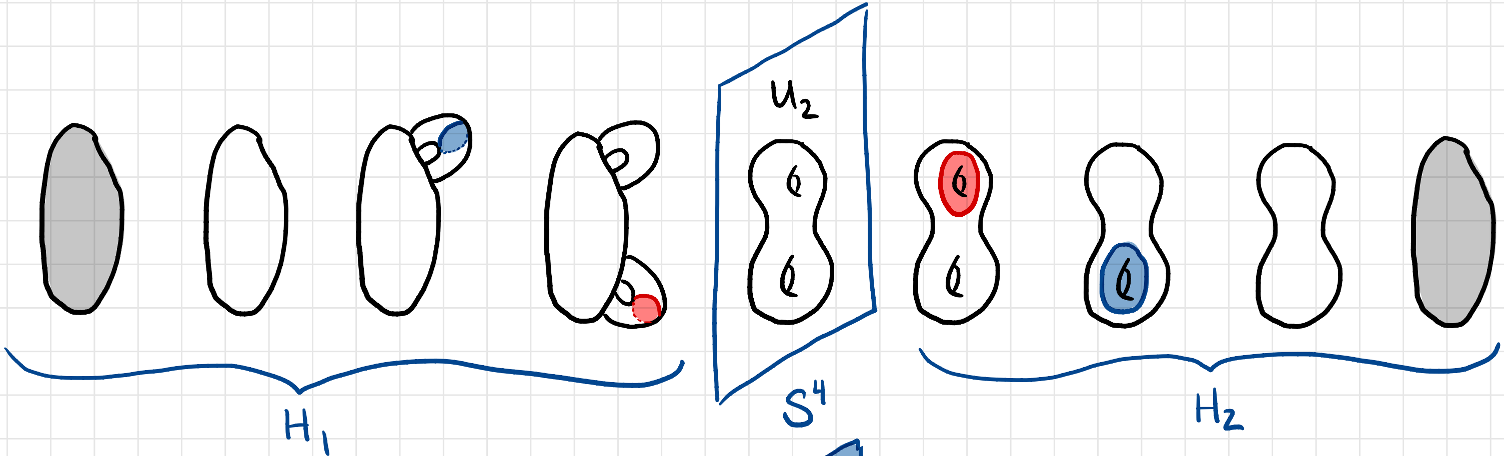
But U_2 also bounds standard handlebodies $\tilde{H}_1$ and $\tilde{H}_2$ in S^4:





Push $\tilde{H}_1$ and $\tilde{H}_2$ into B^5 on opposite sides of S^4:





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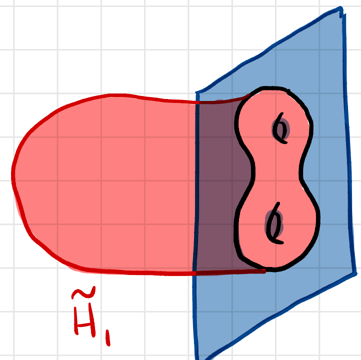

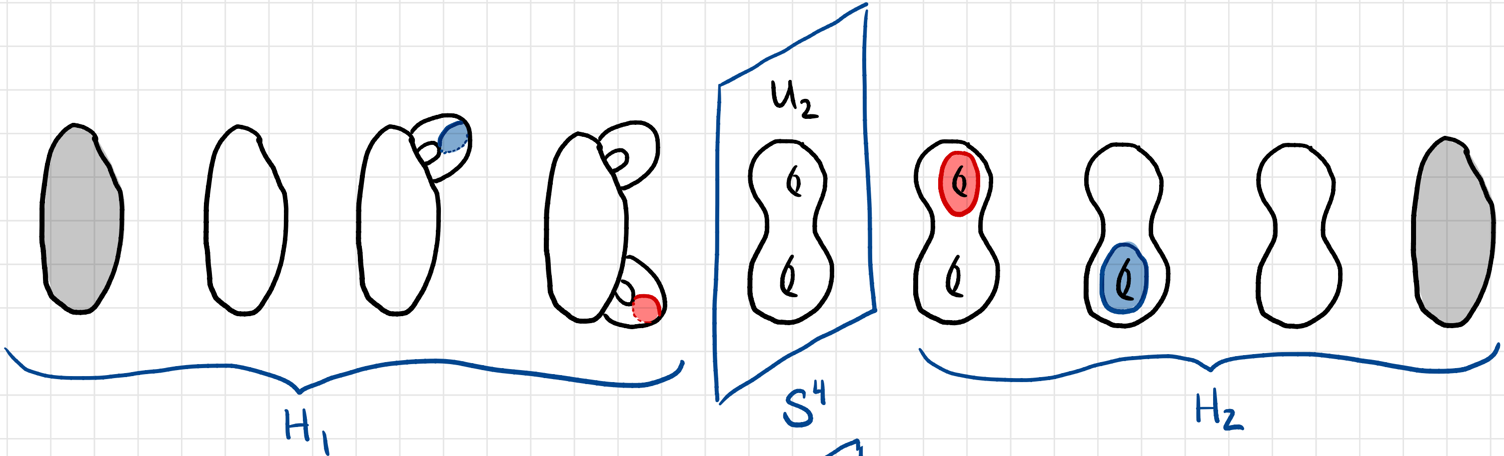

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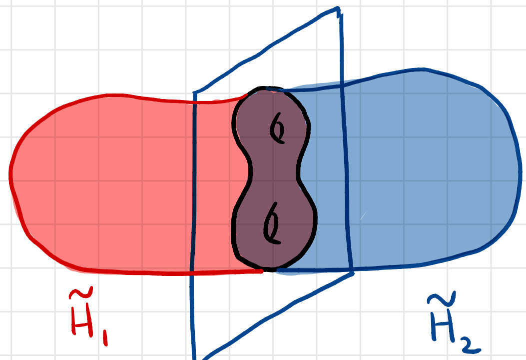


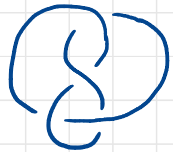
$\tilde{H}_1 \cup \tilde{H}_2$ is an unknotted $S^3 \subseteq S^5$.

not isotopic

Since $P = H_1 \cup H_2$ is knotted, either $H_1 \not\simeq \tilde{H}_1$ or $H_2 \not\simeq \tilde{H}_2$. $\square$

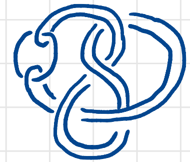
Bing doubles and split links

Knots in S^3:



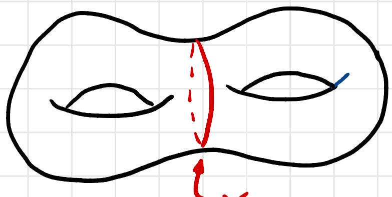
K

$\xrightarrow[\text{double}]{\text{Bing}}$



B(K)

For knots in S^n:

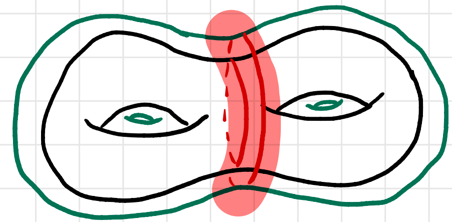


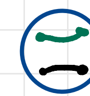
Σ^{n-2}

$\gamma \cong S^{n-3}$

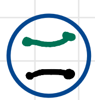
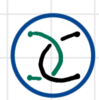
Separating

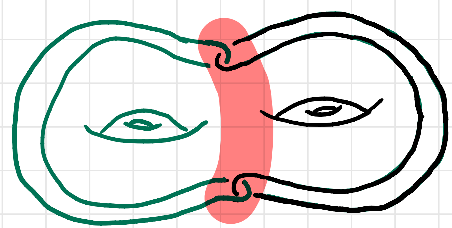
$\xrightarrow[\text{copy of } \Sigma]{\text{Parallel}}$



$\gamma \times B^3 = \gamma \times$ 

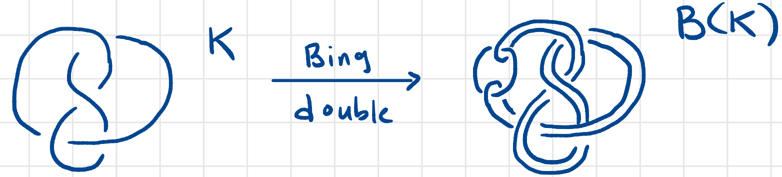
$\xrightarrow{\text{Tangle replacement}}$

$\gamma \times$  $\longmapsto$ $\gamma \times$ 



Bing doubles and split links

Knots in S^3:



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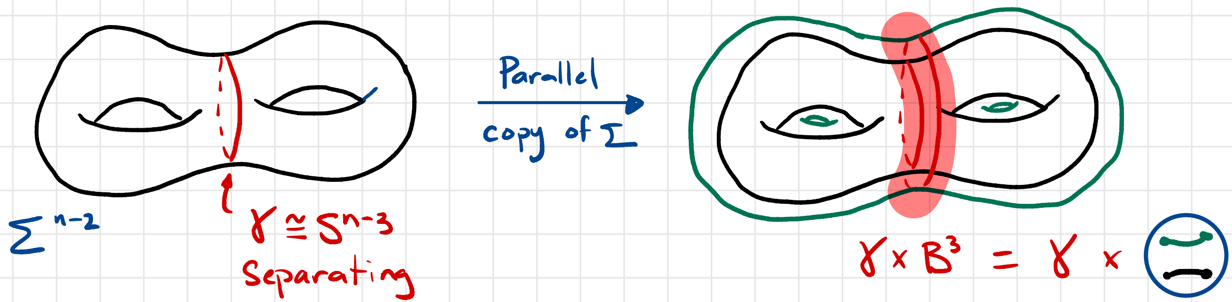
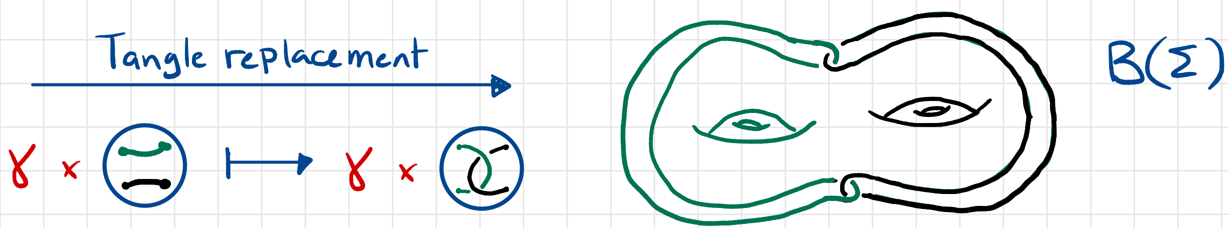
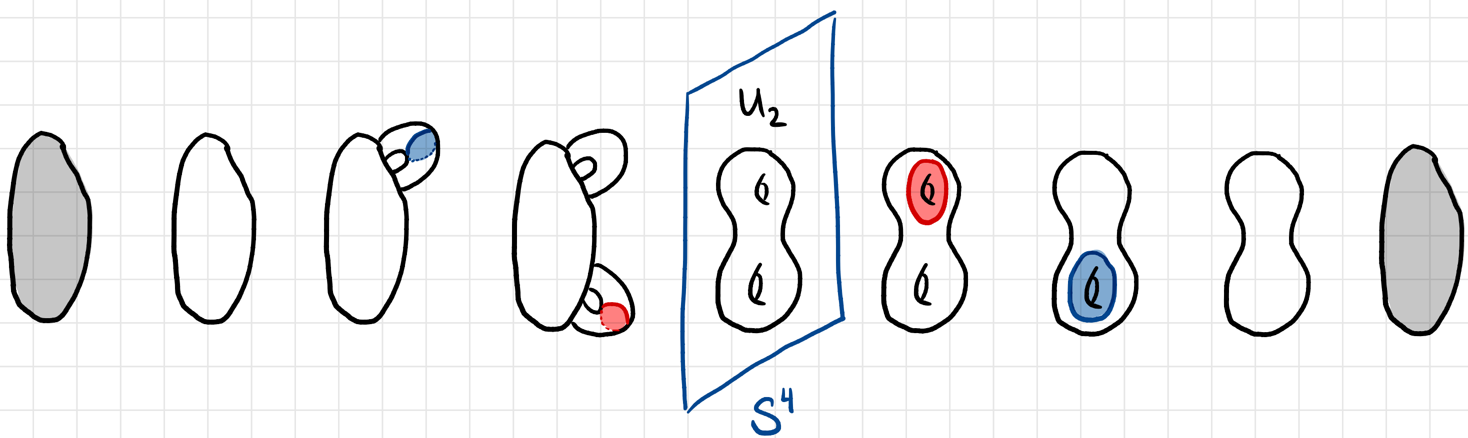


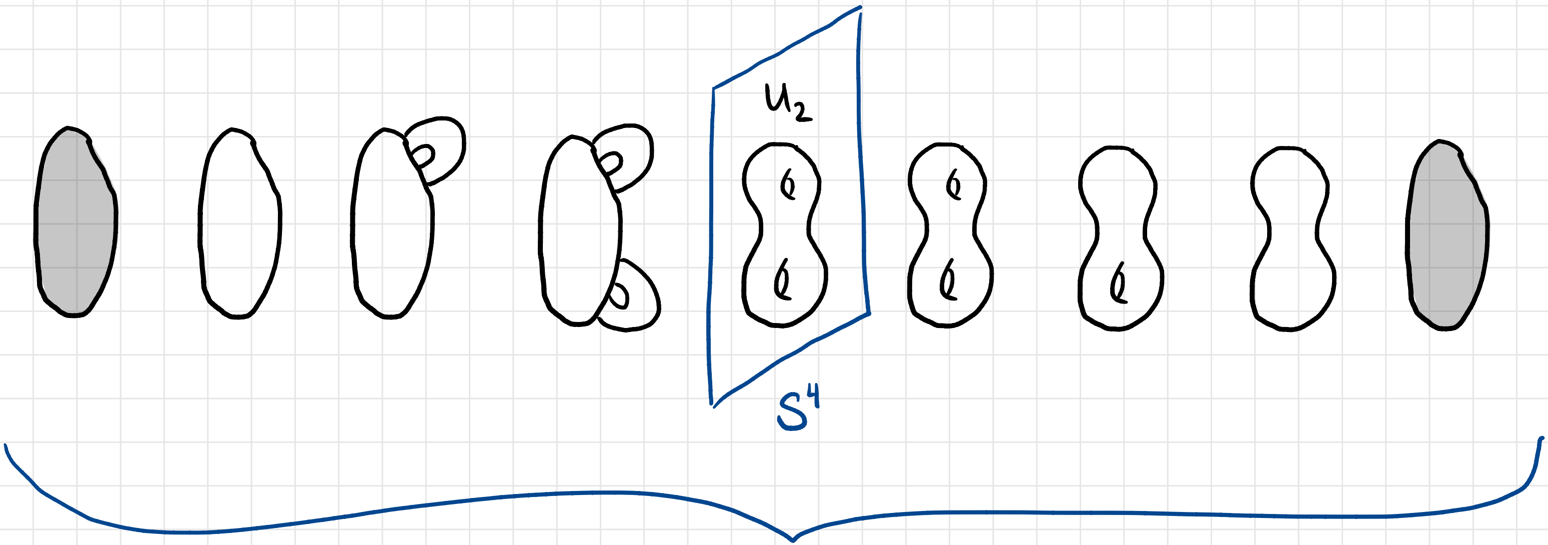


Lemma: If $U_n \subseteq S^4$ is unknotted, then $B(U_n)$ is a smooth unlink for any choice of γ.

Lemma: If P is a non-trivial 3-knot, then $B(P)$ is not split.

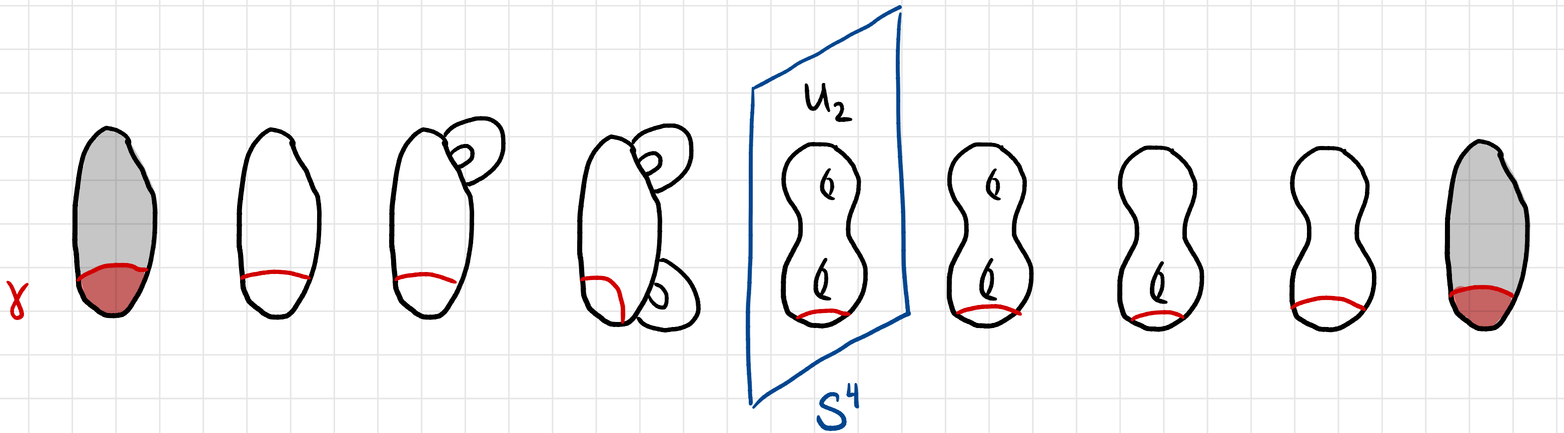
Strategy: Take Bing double of the knotted 3-sphere P from before:

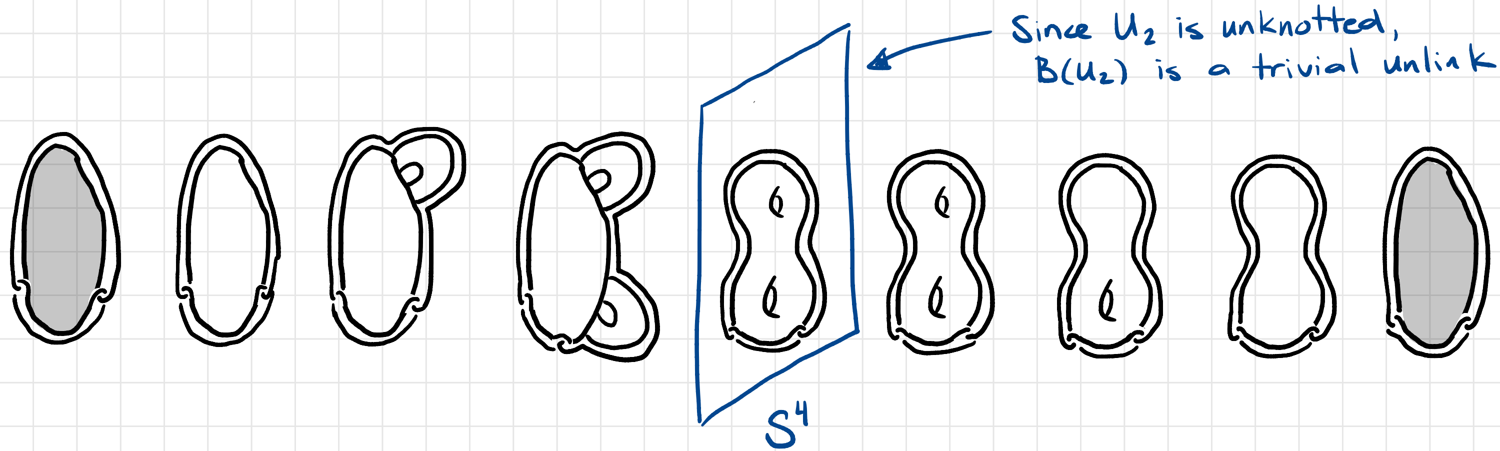


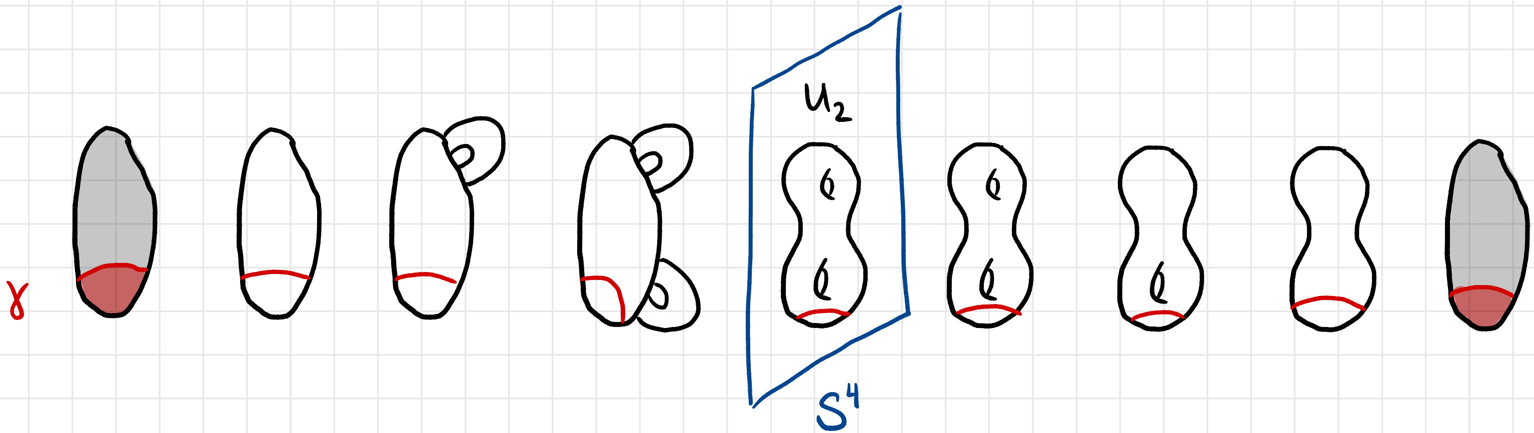


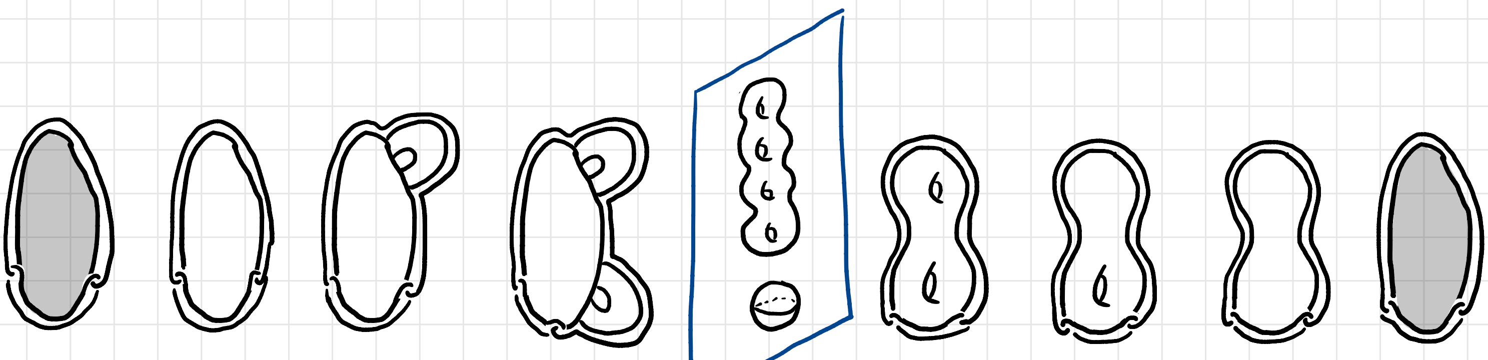
P knotted S^3 in S^5

with unknotted central cross-section U_2 in S^4.

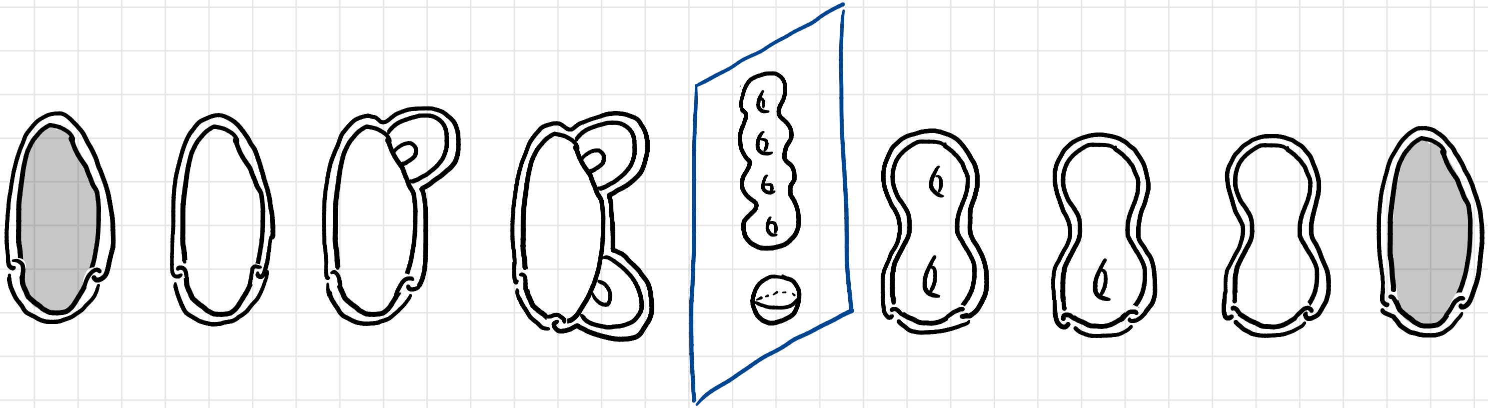

γ
U2
S4


Since U2 is unknotted,
B(U2) is a trivial unlink
S4



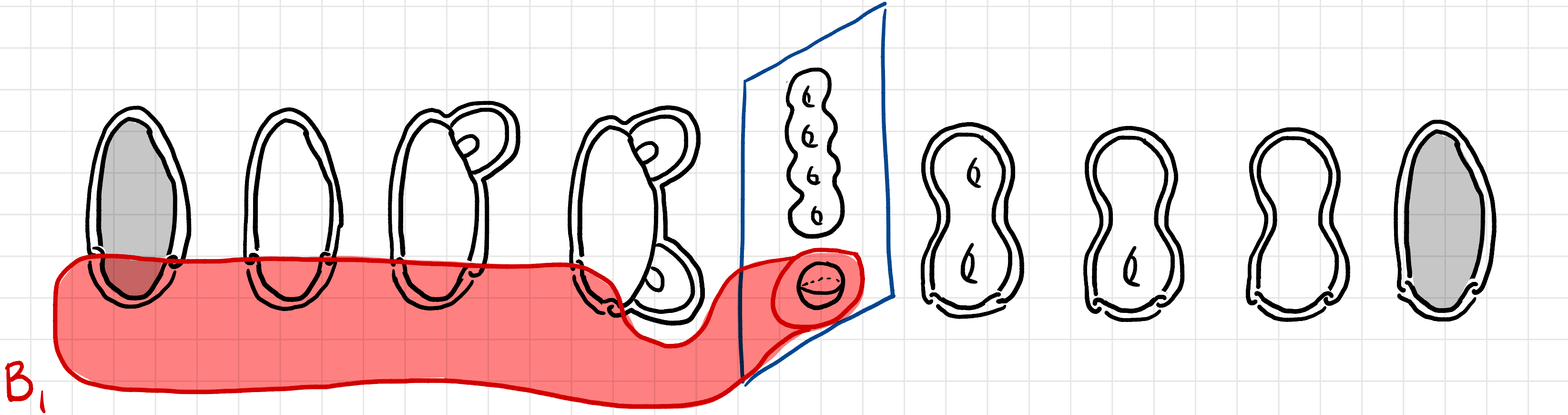


Note: Since P was knotted, $B(P)$ is <u>not</u> a split link.



Both halves of B(P) are boundary parallel (not simultaneously though).

Pushing each half into S^4 separately, we obtain 4-balls B_1 and B_2 which split their respective halves.



Both halves of $B(P)$ are boundary parallel (not simultaneously though).

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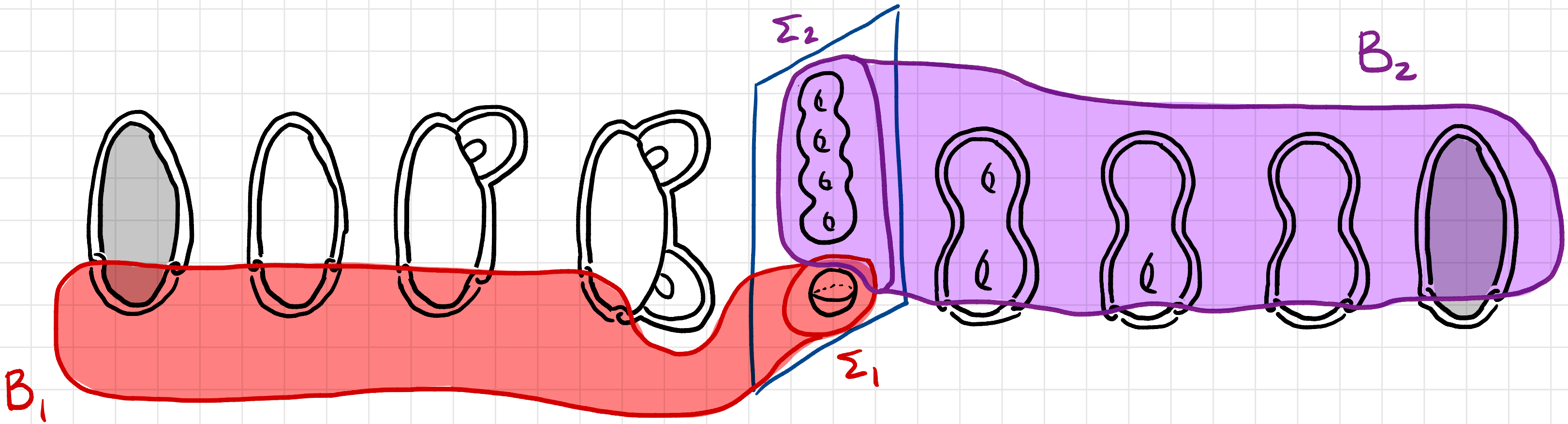

Both halves of $B(P)$ are boundary parallel (not simultaneously though).

Pushing each half into S^4 separately, we obtain 4-balls B_1 and B_2 which split their respective halves.

Set $\Sigma_1 = \partial B_1$ and $\Sigma_2 = \partial B_2$.

Both Σ_1 and Σ_2 are splitting spheres for the central cross-section.

Σ_1 and Σ_2 can't be isotopic, otherwise $B_1 \cup B_2$ could be used to build a splitting sphere for $B(P)$, which is not split. $\square$

Thank you!