Corks for exotic diffeomorphisms

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Based on joint work with
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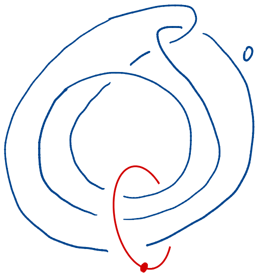
Warm-up: Corks for exotic 4-manifolds



The Akbulut-Mazur manifold. There is a diffeomorphism (involution) of the boundary which does not extend to a smooth diffeomorphism of the 4-manifold.

Cutting it out of a closed 4-manifold and re-gluing it using the involution may produce an exotic smooth structure. (Gauge theory or Floer theory)

The cork theorem for h-cobordisms

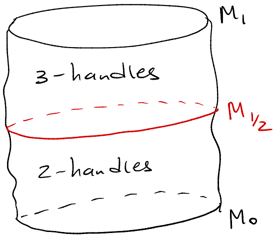
Curtis-Hsiang-Freedman-Stong '95
Matveyev '95

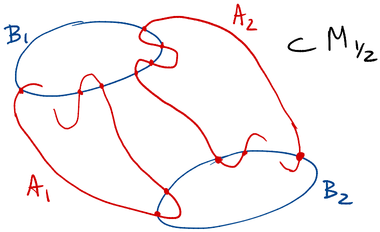
Let W be a smooth 5-dimensional h-cobordism between two simply connected, closed 4-manifolds, X_0, X_1.

Then there exists a *contractible* sub-h-cobordism $C \subset W$ between $C_0 \subset X_0$ and $C_1 \subset X_1$ such that $W \setminus C$ is a product h-cobordism.

Here C_0, C_1 can be shown to satisfy additional properties (which are usually assumed when using the term "cork"), such as the existence of an invlution on the boundary.

The proof involves analysis of a handle structure of the "middle level" $M_{1/2}$ starting from the attaching spheres $\{B_i\}$ of the 3-handles and the belt spheres of the 2-handles





Since homeomorphic simply-connected smooth 4-manifolds are smoothly h-cobordant, it follows that exotic smooth structures on such manifolds are related by cork twisting.

Mapping class groups of closed simply-connected 4-manifolds

Classification of *homeomorphisms* up to isotopy:

Theorem [Quinn, 1986]

$$\pi_0 \operatorname{Homeo}^+(M) \xrightarrow{\cong} \operatorname{Aut}(H_2(M), \lambda_M),$$

sending an isotopy classes of orientation-preserving homeomorphisms of M to the induced isometry of the intersection pairing $\lambda_M \colon H_2(M) \times H_2(M) \to \mathbb{Z}$, is an isomorphism.

Building on work of Mike Freedman '82

A correction:
Gabai-Gay-Hartman-K.-Powell '23

Related work: Kreck '79, Perron '86

Mapping class groups of closed simply-connected 4-manifolds

Stable smooth isotopy:

If two self-diffeomorphisms f, g of a simply-connected smooth 4-manifold M are topologically isotopic, they are stably smoothly isotopic.

$f \# \mathrm{Id}, g \# \mathrm{Id} \colon M \#_k S^2 \times S^2 \to M \#_k S^2 \times S^2$ are smoothly isotopic, for some k.

Quinn '86 (corrected by GGHKP '23)
Gabai '22

Mapping class groups of closed simply-connected 4-manifolds

Exotic *diffeomorphisms*:

Topologically but not smoothly isotopic.

First examples: Ruberman '98

Later examples:
Kronheimer-Mrowka '20, Baraglia-Konno '20, J. Lin '20, Konno-Mallick-Taniguchi '23, and others.

Theorem 1 (KMPW)

Let X be a compact, simply-connected, smooth 4-manifold and let $f\colon X \to X$ be a self-diffeomorphism such that

$$f \# \mathrm{Id}\colon X\#(S^2 \times S^2) \to X\#(S^2 \times S^2)$$

is smoothly isotopic to the identity and $f|_{\partial X} = \mathrm{Id}_{\partial X}$.

Then there exists a smooth, contractible, compact codimesnion zero submanifold $W \subseteq X$, such that f is smoothly isotopic to a diffeomorphism supported on W, i.e. that is the identity on $X \setminus \mathring{W}$.

We call W a *diff-cork* for f.

Theorem 2 (KMPW)

Let X be a smooth, compact, simply-connected 4-manifold, and let $f\colon X \to X$ be a diffeomorphism such that f is topologically isotopic to identity.

Then there exists a smooth, compact, codimension zero submanifold $\mathcal{B} \subset X$, such that $\mathcal{B} \simeq \vee_n S^2$, $\mathcal{B} \to X$ is null-homotopic, and f is smoothly isotopic to a diffeomorphism supported on $\mathcal{B}$.

The underlying method: pseudo-isotopy theory (Higher dimensions: Cerf, Hatcher-Wagoner, Igusa, ...)

X a smooth, compact 4-manifold. A *pseudo-isotopy* on X is a diffeomorphism

$$F\colon X \times [0,1] \xrightarrow{\cong} X \times [0,1]$$

that restricts to the identity on $X \times \{0\} \cup \partial X \times [0,1]$.

Kreck ('79) classified diffeomorphisms up to smooth pseudotisotopy. It follows that two topologically isotopic diffeomorphisms $f, g\colon X \to X$ are smoothly pseudoisotopic.

Cerf graphic

Given a pseudo-isotopy $F\colon X \times I \to X \times I$, we can define a 1-parameter family of generalized Morse functions $f_t\colon X \times [0,1] \to [0,1]$ and gradient-like vector fields, where $f_0 = \mathrm{pr}_2\colon X \times I \to I$ and $f_1 = \mathrm{pr}_2 \circ F$.

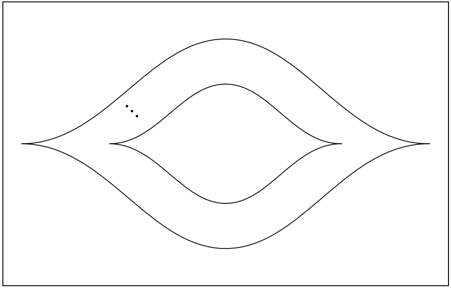
Both f_0 and f_1 have no critical points. There is a generic 1-parameter family of generalized Morse functions (a *Cerf family*) $f_t\colon X \times I \to I$, interpolating between f_0 and f_1. Here isolated degenerate critical points (order 3 singularities) are allowed.

An approach to proving that pseudo-isotopy implies isotopy consists of deformations of these 1-parameter families. The goal is to deform f_t until there are no critical points for all $t \in [0,1]$. (Then such a pseudo-isotopy F is an isotopy.)

Hatcher-Wagoner '73: a two-stage obstruction theory.

First obstruction $\Sigma(F)$ of a pseudoisotopy F: an element of the secondary Whitehead group $\mathrm{Wh}_2(\pi)$ of a group π.

When $\Sigma(F) = 0$ the 1-parameter family can be deformed so that the Cerf diagram consists of *nested eyes* with critical points of index 2 and 3 only.



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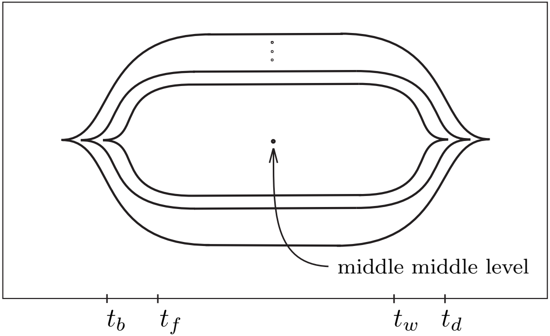
For the trivial group, $\mathrm{Wh}_2(\{1\}) = 0$, so $\Sigma(F) = 0$ in the simply-connected case.

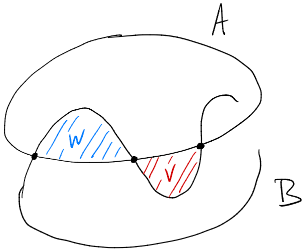
Hatcher-Wagoner, Igusa: (in dimensions ≥ 6) a secondary obstruction

$$\Theta\colon \ker \Sigma \to \mathrm{Wh}_1(\pi_1(M); \mathbb{Z}/2 \times \pi_2(M))/\chi(K_3(\mathbb{Z}[\pi_1(M)]))$$

Quinn's approach is based on the analysis of the middle-middle level



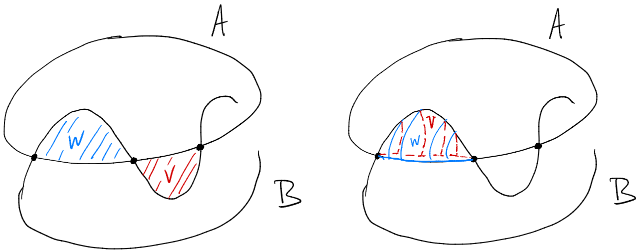
The Quinn core $A \cup B \cup V \cup W$ in the middle-middle level



A: belt spheres of 2-handles
B: attaching spheres of 3-handles
V: finger move disks
W: Whitney move disks

Generally, finger disks intersect Whitney disks!

The Quinn core The union of the boundaries of finger and Whitney disks, both on A and on B, consists of an immersed arc and a collection of immersed circles.



Quinn's arc condition: For a single pair (A, B), the family can be deformed so that there are no circles, and the arc is embedded, both in A and in B.

Theorem (The cork theorem for a 1-eye pseudoisotopy, KMPW)
If F admits a Cerf family with one eye, then there exists a compact, contractible, codimension zero, submanifold $W \subseteq X$ and a smooth isotopy of F, rel. $X \times \{0\} \cup \partial X \times I$, to a pseudo-isotopy F', such that $F' = \mathrm{Id}$ on $(X \setminus \mathring{W}) \times I$.

In our applications we use

Theorem (Gabai '22) Let $f\colon X \to X$ be a diffeomorphism with $f|_{\partial X} = \mathrm{Id}$. Then

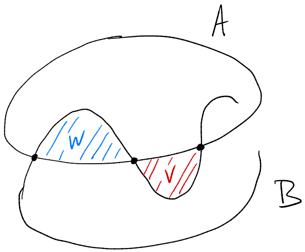
$$f \# \mathrm{Id}\colon X \#^n (S^2 \times S^2) \to X \#^n (S^2 \times S^2)$$

is smoothly isotopic to the identity rel. boundary if and only if f is pseudo-isotopic to the identity via an n-eyed pseudo-isotopy.

Outline of the proof of the cork theorem for a 1-eye pseudoisotopy

- We use Quinn's arc condition. Then the Quinn core $Q = A \cup B \cup V \cup W$ has the homotopy type

$$Q \simeq S^2 \vee S^2 \vee \bigvee^{m} S^1.$$



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• There exists a handle structure $\mathcal{H}$ on $M \setminus Q$, relative to $\partial(M \setminus Q)$, so that:

(1) $\pi_1(Q \cup 1\text{-handles})$ is free, and

(2) there exist 2-handles whose attaching regions represent the conjugacy classes of the free generators of π_1.

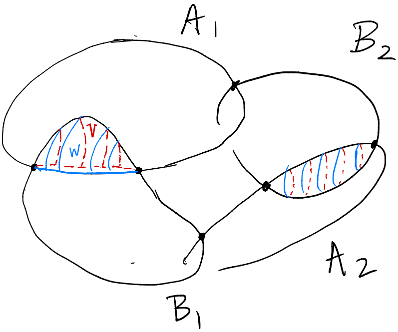
• Attach these 2-handles to make Q simply connected. Attach the two 3-handles (flowing down from A and up from B) to get a contractible h-cobordism.

Theorem 2 (KMPW)

Let X be a smooth, compact, simply-connected 4-manifold, and let $f\colon X \to X$ be a diffeomorphism such that f is topologically isotopic to identity. Then there exists a smooth, compact, codimension zero submanifold $\mathcal{B} \subset X$, such that $\mathcal{B} \simeq \vee_n S^2$, $\mathcal{B} \to X$ is null-homotopic, and f is smoothly isotopic to a diffeomorphism supported on $\mathcal{B}$.

Note: Budney-Gabai's barbell diffeomorphisms are supported in $S^2 \times D^2 \# S^2 \times D^2$.

Outline of the proof of Theorem 2



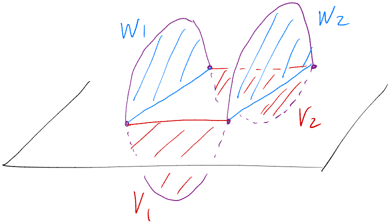
For more then one eye, the union of boundaries of V and W form circles, and the arc condition cannot be achieved. The Quinn core Q has additional π_2 generators formed by the finger and Whitney disks; these generators may be non-trivial in $\pi_2(M)$.

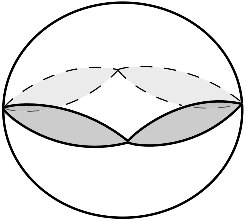
Note that this analysis doesn't come up in the proof of the higher-dimensional (and of the 4-dimensional topological) pseudoisotopy theorems in the simply-connected case.

Outline of the proof of Theorem 2:

There are n eyes, in other words n pairs of 2-spheres (A_i, B_i) in the middle-middle level.

• Generators of π_2 of the Quinn core corresponding to $A_i \cap B_j, i \neq j$:



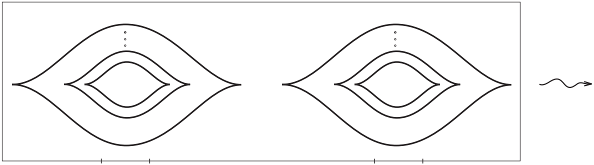


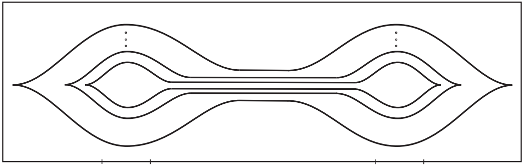
Key ideas:

• Add 2-handles to make the Quinn core simply-connected, as in the 1-eye case.

• Introduce new, *trivial!* pseudo-isotopies, each one realizing a particular π_2 element for $A_i \cap B_j$.

• Concatenate them with the given pseudoisotopy





Examples

First examples of exotic diffeomorphisms of simply-connected 4-manifolds (Ruberman '98, '99):

For $X = \#^{2n}\mathbb{C}P^2\#^{10n+1}\overline{\mathbb{C}P}^2$ if n is odd, and $X = \#^{2n}\mathbb{C}P^2\#^{10n+2}\overline{\mathbb{C}P}^2$ if n is even, there is a subgroup of

$$\ker\left(\pi_0\,\mathrm{Diff}(X) \to \pi_0\,\mathrm{Homeo}(X)\right)$$

which abelianizes to $\mathbb{Z}^\infty$.

These diffeomorphisms are all 1-stably isotopic to the identity (using work of Auckly-Kim-Melvin-Ruberman), so they admit diff-corks.

The original proof that his diffeomorphisms are exotic used Donaldson's theory. Baraglia-Konno more recently (2020) showed that these diffeomorphisms, as well as many others, can be detected using Seiberg-Witten invariants.

Corollary. There exists a compact, contractible, smooth 4-manifold W and a diffeomorphism $\tilde{f}\colon W \to W$, restricting to Id on the boundary, such that $\mathrm{FSW}(\tilde{f}) \neq 0$.

Consider Ruberman's example $f\colon X \to X$, and apply J. Lin's gluing formula

$$FSW(f, \widetilde{\mathfrak{s}}) = \langle FSW(\tilde{f}), SW(X \setminus \mathring{W}) \rangle$$

$\langle\ ,\ \rangle\colon \widehat{HM}_*(\partial W) \otimes \overrightarrow{HM}^*(\partial W) \to \mathbb{Z}$ is the pairing of the monopole Floer homology and co-homology of ∂W.

Consider a compact, contractible n-manifold W. Fix an embedding $D^n \hookrightarrow \mathring{W}$, and let $E_n \colon \mathrm{Diff}_\partial(D^n) \to \mathrm{Diff}_\partial(W)$ be the map given by extending diffeomorphisms of D^n by identity over $W \setminus D^n$. Galatius and Randal-Williams '23 showed that E_n is a homotopy equivalence for $n \geq 6$. (Krannich-Kupers' 24: also for $n = 5$.)

Our next result shows that E_4 need not be a homotopy equivalence.

Theorem Let $(W, \widetilde{f})$ be as in the previous theorem. Then f is not smoothly isotopic to a diffeomorphism supported on the interior of a 4-ball, and thus the map $E_4 \colon \mathrm{Diff}_\partial(D^4) \to \mathrm{Diff}_\partial(W)$ is not surjective on path components, so is not a homotopy equivalence.

Theorem:

For each $m \geq 1$ there exists a contractible, compact, smooth 4-manifold C_m and a collection $\{g_1, \ldots, g_m\}$ of boundary-fixing diffeomorphisms of C_m that generate a subgroup of $\pi_0 \operatorname{Diff}_\partial(C_m)$ that abelianizes to $\mathbb{Z}^m$.

(Konno-Mallick '24 proved that localizing cannot produce *infinite* rank subgroups.)

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Questions

- Is there a contractible diff-cork for a diffeomorphism f which is isotopic to Id after $\#_n S^2 \times S^2$, $n > 1$?
- The Dehn twist on $K3 \# K3$?
- $\pi_1 \neq \{1\}$?