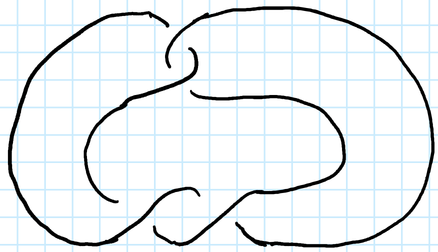
Knots – Online Seminar

(joint w/ A. Daemi + M. Miller Eismeier)

I. Dehn Surgery

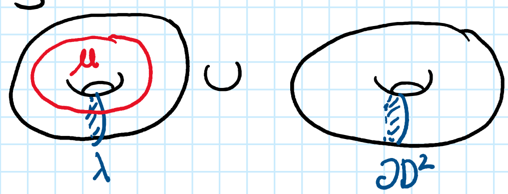
Def: Let $K \subseteq S^3$



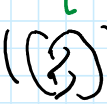
Define $S^3_{\frac{p}{q}}(K) = (S^3 \setminus \nu K) \cup D^2 \times S^1$ $\quad (\gcd(p,q)=1)$

$p\mu + q\lambda \longleftarrow \!\!\shortmid\; \partial D^2$

Ex: ① $S^3_{\frac{1}{0}}(K) = S^3$

② $S^3_{\frac{p}{q}}(O) =$  $= L(p,q)$

Note: $S^3_{\frac{p}{q}}(O) \cong S^3_{\frac{p}{q'}}(O)$ if $q^{\pm 1} \equiv q' \pmod p$

③ $S^3_{-1}($  $) = \Sigma(2,3,7) = S^3_{+1}($  $)$

Rmk: Attaching an n-framed 2-handle to S^3 gives cobordism from S^3 to $S^3_n(K)$.

Q: How injective is surgery w.r.t. p/q?

Thrm: (Gordon–Luecke) $S^3_{\frac{p}{q}}(K) = S^3_{\frac{1}{0}}(K) \Rightarrow K = O$

($\Rightarrow$ knots are determined by their complements)

Conj: (Cosmetic Surgery Conj) If $S^3_{\frac{p}{q}}(K) \cong_+ S^3_{\frac{p'}{q'}}(K)$, then $K = O$ or $\frac{p}{q} = \frac{p'}{q'}$.

What's known?

Suppose $S^3_{\frac{p}{q}}(K) = S^3_{\frac{p'}{q'}}(K)$ and $K \neq O$ and $\frac{p}{q} \neq \frac{p'}{q'}$:

① $H_1(S^3_{\frac{p}{q}}(K)) = \mathbb{Z}/p \Rightarrow |p| = |p'|$.

② As $q \to \infty$, $\mathrm{vol}(S^3_{\frac{p}{q}}(K)) \to \mathrm{vol}(S^3 \setminus K) \Rightarrow$ finitely many $\frac{p'}{q'}$ to consider.

(Futer-Purcell-Schleimer) Using lengths of slopes $\Rightarrow$ for given knot, can enumerate possible $\{\frac{p}{q}, \frac{p'}{q'}\}$

③ (Heegaard Floer homology)

$Y^3 \rightsquigarrow \widehat{HF}(Y) = $ v. space

$\dim \widehat{HF}(S^3_{\frac{p}{q}}(K)) \approx |p| + |q| C(K)$ ← $\neq 0 \Leftrightarrow K \neq O$

(Ozsváth-Szabó, Wang, Ni-Wu) $\frac{p'}{q'} = \frac{-p}{q}$

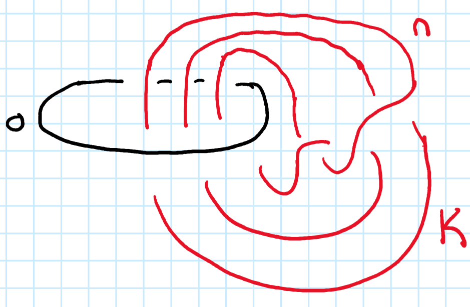
(Hanselman) Using gradings $\{\frac{p}{q}, \frac{p'}{q'}\} = \{+\frac{1}{n}, -\frac{1}{n}\}$ or $\{+2, -2\}$

Unfortunately, $\widehat{HF}(S^3_1(9_{44})) \cong \widehat{HF}(S^3_{-1}(9_{44}))$. :(

<u>Thrm</u>: (Daemi-L.-Miller Eisenmeier)① If $S^3_{\frac{p}{q}}(K) \cong_+ S^3_{\frac{p'}{q'}}(K)$ and $\frac{p}{q} \neq \frac{p'}{q'}$ and $K \neq O$, then $\{\frac{p}{q}, \frac{p'}{q'}\} = \{+2, -2\}$. Further, $\underbrace{g(K) = 2}_{\text{Hanselman}}$ and $\Delta_K = 1$.

$\Rightarrow$ CSC holds if K is fibered or alternating or not top. slice.

② Let $K \subseteq S^2 \times S^1$ gen π_1 w/ $K \neq pt \times S^1$



$Y_n \not\cong_+ Y_m$ for $n \neq m$

($\Rightarrow$ associated Mazur mflds are all diff't)

Idea: Use instanton Floer homology I_* which has special filtration CS

Show CS-filtered I_*'s of $S^3_{\frac{1}{n}}(K)$ and $S^3_{-\frac{1}{n}}(K)$ are different.

Thm: (Daemi-L.) If $b_1(Y) = 0$ + $\chi_{SU(2)}(Y)$ is Morse-Bott, then nullhtpc knots are determined by their complements.

II. Instanton Floer homology (simplified)

Setup: $Y = \mathbb{Z}H_* S^3$

$\chi(Y) = \mathrm{Hom}^{\neq 0}(\pi_1(Y), SU(2))/_{\text{conj.}}$ $\left(\{\text{flat irred. } SU(2)\text{-conn's}\}/_{\text{gauge}}\right)$

$CS: \chi(Y) \times \mathbb{Z} \to \mathbb{R}$ $\left(CS(\alpha) = \frac{1}{8\pi^2}\int_Y \mathrm{Tr}(\alpha \wedge d\alpha + \frac{2}{3}\alpha\wedge\alpha\wedge\alpha)\right)$

$I_*(Y) \approx H_*$ of chain cplx gen'd by $\chi(Y) \times \mathbb{Z}$
filtration by CS

$\left(I_*(Y) \text{ is Morse } H_* \text{ of } CS: \Omega^1(Y; su_2)/_{\substack{\deg 0 \\ \text{gauge} \\ \text{transf's}}} \to \mathbb{R}\right)$

Ex: $I_*(S^3_{-1}(4_1)) = \mathbb{Z}[t,t^{-1}]\langle \alpha_{(1)}, \alpha_{(5)} \rangle$, t raises grading by 8, CS by 1

$CS = -\frac{121}{168}$ $\quad CS = -\frac{25}{168}$ (mod 1)

$I_*(S^3_{+1}(3_1)) = \mathbb{Z}[t,t^{-1}]\langle \beta_{(1)}, \beta_{(5)} \rangle$

$CS = \frac{1}{120}$ $\quad CS = \frac{49}{120}$ (mod 1)

I_*'s are same, CS filtrations are diff't!

Functoriality: If $W: Y \to Y'$ is a cobordism, we get $I(W): I(Y) \to I(Y')$

$(\deg = -3(b_1(W)+b^+(W)))$

$$I(W)(\alpha) = \sum \underbrace{\#\mathcal{M}(\alpha,\beta)}\,\beta$$

moduli space
of ASD conn's on W
w/ limits to A,B on ends

If $\mathcal{M}(\alpha,\beta) \neq 0$, then either ① $CS(\alpha) > CS(\beta)$ or

② $CS(\alpha) = CS(\beta)$ and $\exists \rho: \pi_1(W) \to SU(2)$ extending α, β

Metathrm: If $W: Y \to Y'$ induces iso. on I_*, $\pi_1(W) = 0$, $b^+(W) = 0$, and $I_*(Y') \neq 0$, then $Y \not\cong_+ Y'$.

Proof: $I(W): I(Y) \xrightarrow{\cong} I(Y')[\text{strict filtration shift}] \not\cong I(Y')$. $\square$

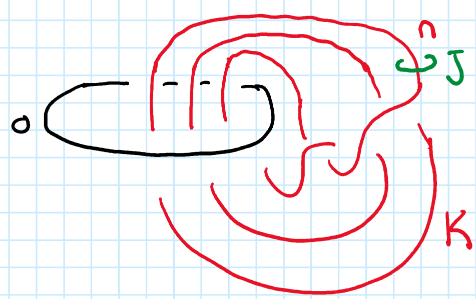
How to show cobordism map is isomorphism?

Exact triangles! $\exists$ exact triangle

$$\begin{tikzcd} I(Y) \arrow[rr, "I(\text{-1-framed 2-handle attach})"] & & I(Y_{-1}(K)) \arrow[dl] \\ & I(Y_0(K)) \arrow[ul] & \end{tikzcd}$$

(counts certain non-abelian SO(3) representations)

Thrm ②: Let $K \subseteq S^2 \times S^1$ gen π_1 w/ $K \neq pt \times S^1$



$Y_n \not\cong_+ Y_m$ for $n \neq m$

Proof: (Show $Y_{n+1} \not\cong Y_n$) Let $W: Y_n \to Y_{n+1}$ be -1-framed 2-handle attached along J. $\pi_1(W) = 0$ and $b^+(W) = 0$

Exact triangle for J: $\begin{tikzcd} I(Y_n) \arrow[rr, "I(W)"] & & I(Y_{n+1}) \arrow[dl] \\ & \cancel{I'(S^2 \times S^1)} \arrow[ul] & \end{tikzcd}$ $\Rightarrow I(W) = \text{iso.}$

(L.-Pinzón-Caicedo-Zentner) $I(Y_{n+1}) \neq 0$

Metathm $\Rightarrow Y_n \not\cong_+ Y_{n+1}$. $\square$