bounding the Dehn surgery number by 10/8

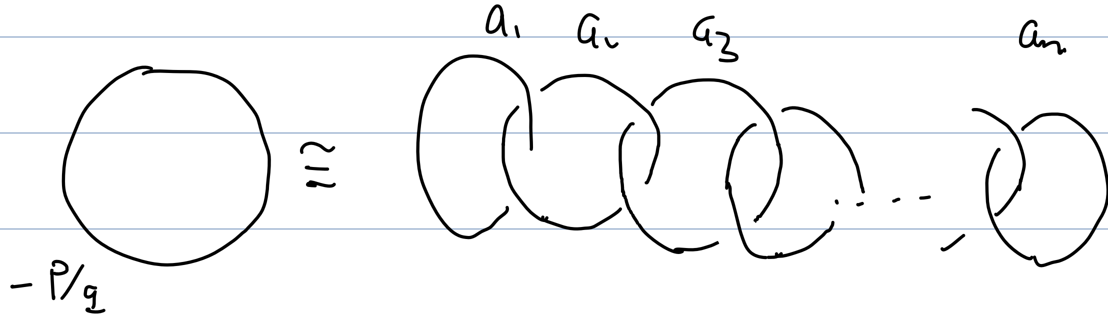
j. with L. piccirillo

Motivation to study links in S^3

Thm (Lickorish – wallace) Every closed, oriented, connected 3-mfd M is obtained by Dehn surgeries on links in the three-sphere.

$$M = S^3_{d_1, d_2 \cdots d_n}(L) \qquad d_i \in \mathbb{Q}.$$

L(p, q)



$[a_1 \cdots a_n]$ is a continued fraction expansion of $-\frac{p}{q}$, i.e.

$$-\frac{p}{q} = a_1 - \cfrac{1}{a_2 - \cfrac{1}{a_3 - \cfrac{1}{a_4 - \cdots \frac{1}{a_n}}}}$$

Q: Given a closed, oriented 3-mfd M, what's the minimal number of links required to obtain M by Dehn surgeries?

Q1: Find M that is not an surgery on a knot in S^3.

$$M \neq S^3_d(K)$$

Q2: Given $n \geq 2$. Find M_n such that M_n is not a surgery on any n-component links L_n?

Possible obstructions:

$M \neq S^3_d(K)$

Q1

① Casson invariant (Boyer-Lines homology lens space)

② Taubes' end-periodic diagonalization theorem (Auckly homology sphere)

③ Heegaard Floer homology (homology sphere Kim-Karakurt-Lidman)

④ Rohlin invariant / Heegaard Floer homology (homology $S^1 \times S^2$ Hedden-Kim-Mark-Park)

⑤ SU(2) character variety (homology lens space Sivek-Zentner)

(homology sphere examples are particularly interesting)

Q2: Daemi-Miller-Eismeier announced

$\#^n PHS^3$ requires the surgery link to have sufficiently large components. Saying at least n-component.

Our obstruction: Furuta's 10/8-theorem.

Theorem. (Furuta) Let X be a closed, spin, smooth 4-mfd with $b_2(X) \neq 0$. and indefinite intersection form. Then

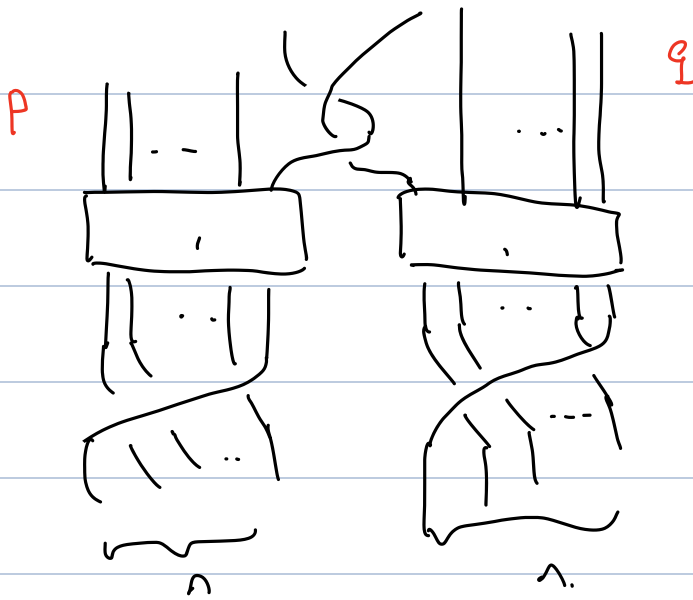
$$4b_2(X) \geq 5|\sigma(X)| + 8$$

Second Betti number of X Signature of X.

Def: X^n is a spin mfd is the tangent bundle TX admits a spin-structure, i.e. a double cover of the principal $SO(n)$-bundle over X.

Remark: $w_2(X) \in H^2(X;\mathbb{Z}_2)$ is an obstruction for spin-structure on X.

<u>Theorem</u> (LP) For any odd integer p, q, and odd integer n, sufficiently large, the surgery manifold $S^3_{p,q}(L_n)$ is not a surgery on a knot in S^3



L_n

2-component link with each component $T(n, n+1)$

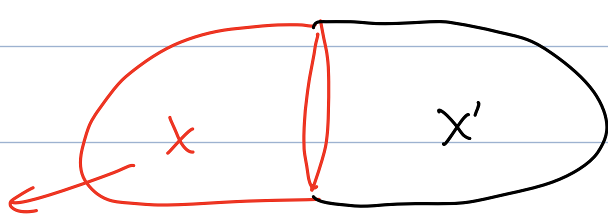
$\underbrace{S^3_{p,q}(L_n)}_{=M}$ is $\mathbb{Q}HS^3$ $\quad |H_1(M)| = pq-1$

<u>Corollary</u>: For any integer k and sufficiently odd integer n, the homology lens space $L(2k,1)$ given by $S^3_{2k+1,1}(L_n)$ is not a surgery on a knot in S^3; $\Rightarrow$ $\#^m S^3_{p,q}(L_n)$ is not surgery on m-component links.

proof: let $p = 2k+1$, $q = 1$.

<u>Idea</u>: suppose $S^3_{p,q}(L_n) = S^3_d(K)$

4-mfd



$\partial X = S^3_d(K)$

$\partial X' = S^3_{p,q}(L_n)$

} s spin structure on M: $Spin(L)$

requires some spin-structure on $X \cup X'$ so that we will get

a contradiction to Furuta's 10/8-theorem.

spin-structure on 3-mfd and possible extensions to 4-mfds

Extend spin-structure s to 4-mfds X. X' via Kirby Calculus

Natural choices X: $B^4 \cup_K$ 2-handle $\curvearrowright D^2 \times D^2$

X' : $B^4 \cup_{L_n}$ 2 2-handles

both X and X' are 2-handle bodies (i.e consists of 0- and 2-handles)

also called trace of the link.

<u>Def</u> (Characteristic sublinks) Given a framed link $L = K_1 \cup \dots K_n \in S^3$

a characteristic sublink $L' \subseteq L$ is a sublink such that for each K_i in L,

$$lk(L', K_i) \equiv lk(L_i, L_i) \mod 2.$$

<u>Kaplan</u>: spin-structures s on M $\xleftrightarrow{\text{1-1 correspondence}}$ Characteristic sublinks of L.

extend spin-structure s to X $\xleftarrow[\text{Vanishing } w_2(X, s)]{}$ $w_2(X, s) \in H^2(X, M; \mathbb{Z}_2)$

<u>e.g.</u>: If the empty link is characteristic, then the trace X for L is

a spin filling for (M, s).

If characteristic sublink $L' \neq \emptyset$. Apply the following Kirby moves.

1) Add / remove any ± 1-framed unknot O so that

$$L'' = L' \cup O$$ (blow up / down)

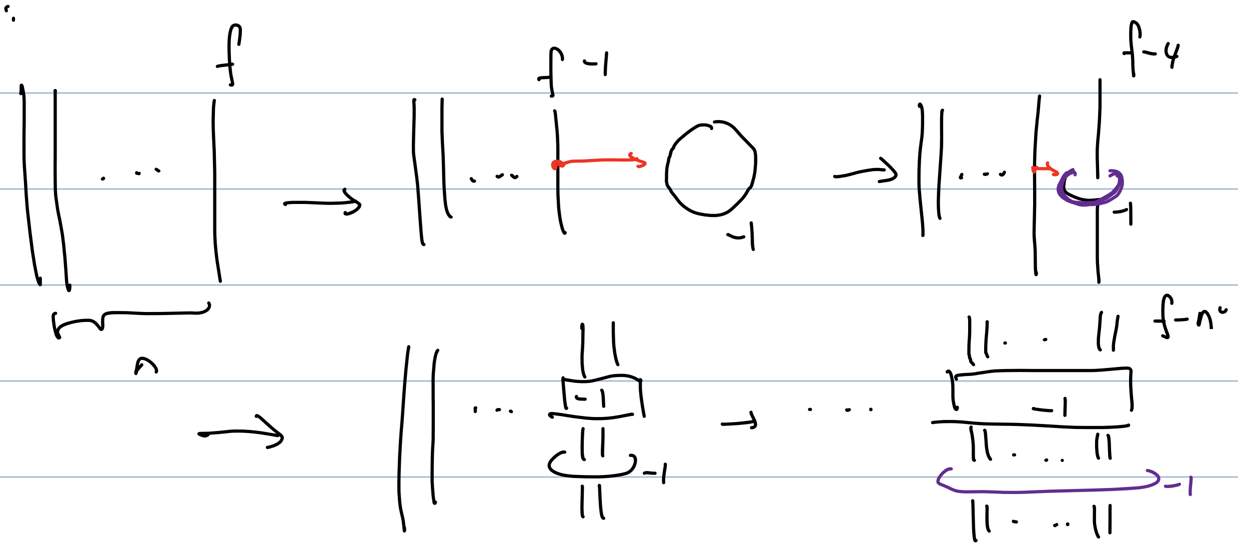
2) band sum: Given $K_1, K_2 \in L'$

$$L'' = (L' \setminus (K_1 \cup K_2)) \cup (K_1 \# K_2)$$ (handle slide)

3) band sum: Given $K_1 \subset L'$ $K_2 \subset L \setminus L'$

$$L'' = (L' \setminus K_1) \cup ((K_1 \# K_2) \cup K_2)$$

Ex:



Proof of Main Theorem: special case 0-surgery

i.e. $pq - 1 = 0 \Rightarrow p = q = 1$ or $p = q = -1$

let's assume $p = q = 1$ and $S^3_{1,1}(L_n) = S^3_0(K)$ for some knot K

For $S^3_0(K)$ 2 spin structure corresponding to the $\emptyset$ empty link and K.

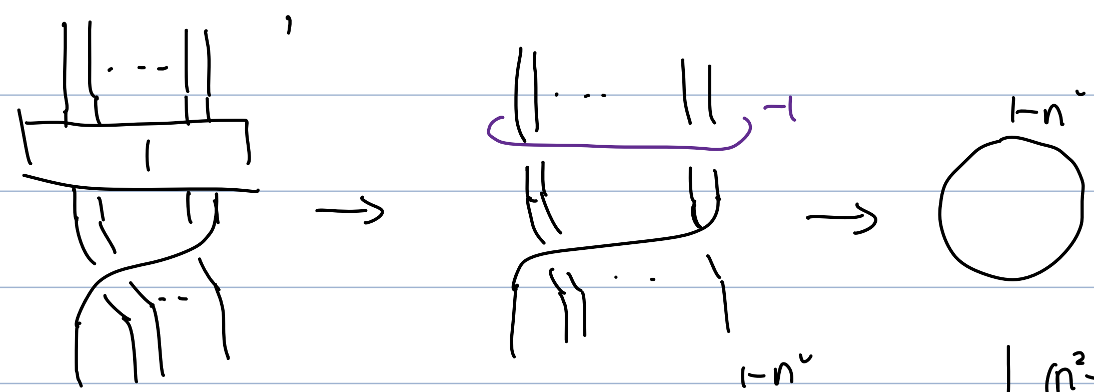
$S^3_{1,1}(L_n)$ 2 spin structure corresponds to $K_1 . K_2$

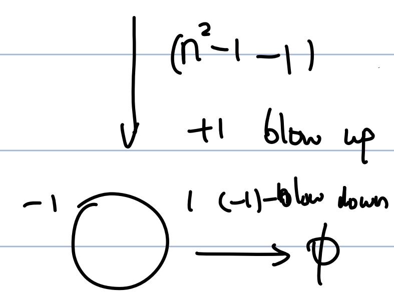
let s be the spin structure on $S^3_0(K)$ corresponding to the empty link and since K_1 and K_2 are symmetric. without loss of generality. assume it corresponds to K.

The trace X for $S^3_0(K)$ is the spin filling for $(S^3_0(K), s)$

On the other hand, we need to construct X' which is the spin-filling of $(S^3_{1,1}(L_n), s)$ . . .

Start with the trace of L_n $\xrightarrow[\text{to kill } K_1]{\text{Apply Kirby moves}}$ spin filling X'.





$b_2(X) = 1$

$\sigma(X) = 0$

$b_2(X') = 2+1 + n^2-1-1 -1 = n^2$

$\sigma(X') = 1 - 1 + n^2-1-1 + 1 = n^2-1$

$b_2(X \cup X') = n^2+1$

$\sigma(X) = n^2-1$

Apply Furuta's 10/8 thm-

$4(n^2+1) \geq 5(n^2-1)+8$

which is a contradiction for $n \geq 2$.

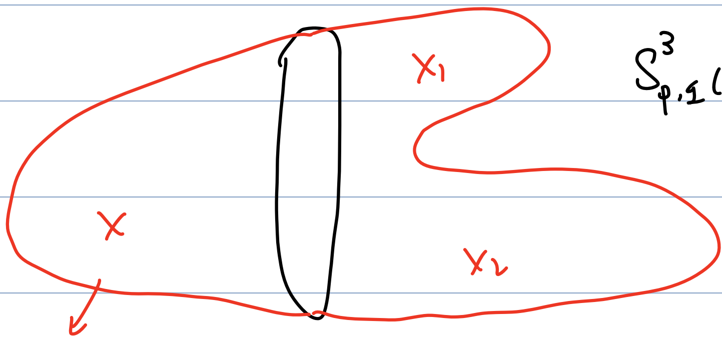
In general goal: $S^3_{p,q}(L_n) \neq S^3_{s/t}(K)$ $s = \pm(pq-1)$ even
$\gcd(s,t)=1$ t odd.

Observation/fact:

$$S^3_{s/t}(K) \# L(t,s) = S^3_{st}(K_{t,s})$$

$\parallel$ ↑ (t,s)-cable of K

$$S^3_{p,q}(L_n) \# L(t,s) = S^3_{ts}(K_{t,s})$$



$\partial X_1 = S^3_{s/t}(K) = S^3_{p,q}(L_n)$

$\partial X_2 = L(t,s)$

$\partial X = S^3_{st}(K_{t,s})$ $S^3_{st}(K_{t,s})$ $(M, \mathfrak{s})$

st is even, $\emptyset$ link is a characteristic sublink

let $\mathfrak{s}$ be the spin structure.

$b_2(X) = 1$ $b_2(X) = \pm 1$

Claim: Again, X_1 has two spin structure which correspond to K_1 and K_2 respectively.

Q: Control spin-filling for $L(t,s)$ for arbitrary t.

(here s is fixed, but t can be arbitrarily large)

Lemma: For lens space $L(t,s)$ with s even, $\gcd(t,s)=1$.

$L(t,s) =$

a_i are even for all $1 \le i \le k$, $k \le |s|$

In fact $L(t,s)$ has only 1 spin structure, which corresponding to the empty link. $\rightsquigarrow$ Spin filling X_2 with $G_2(X_2) \leq |S|$

$|G_2(X_2)| \leq |S|$.