The Wiegold problem and free products of left-orderable groups.

Yash Lodha (joint with Lvzhou Chen)

Purdue University

February 19, 2026

The Adian-Rabin theorem.

Theorem

There does not exist an algorithm that, given a finite presentation $G = \langle S \mid R \rangle$, decides whether or not the group defined by this presentation is non-trivial.

More generally, a property P of groups is a *Markov property* if:

1. There exists a finitely presentable group with property P.
2. There exists a finitely presentable group that cannot be embedded as a subgroup in any finitely presentable group with property P.

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Let P be a Markov property. There does not exist an algorithm that, given a finite presentation $G = \langle S \mid R \rangle$, decides whether or not the group defined by this presentation has P.

Ex: trivial, finite, torsion-free, residually finite, simple, amenable...

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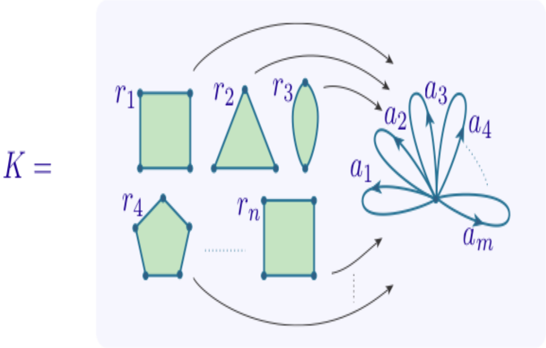
Groups as Topological objects!

The Cayley 2-complex of a group presentation:



(picture credit: Tim Riley's blog "Here there be dragons")

Viewing groups as geometric objects!

Endow a group G with a finite generating set S with an intrinsic geometric structure!

Definition The Cayley graph $\Gamma_S(G)$:

Vertices:$=G$ Edges:$=\{(f, fh) \mid f \in G, h \in S\}$.

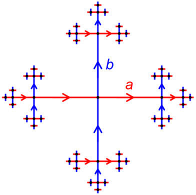


Figure: The Cayley graph of the free group of rank 2.

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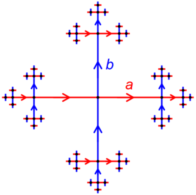

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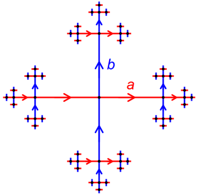

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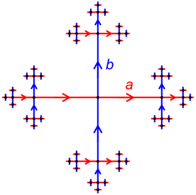

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Endowed with the *word metric*: $d(f, g) :=$ the length of the shortest path connecting f, g in $\Gamma_S(G)$.

The group acts isometrically on any of its Cayley graphs, by left multiplication.

From this metric various notions of *Geometric properties of groups* naturally emerge: For example, Hyperbolic groups!

The mantra of Geometric group theory Study groups as geometric objects!

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Viewing groups as dynamical objects!

- Groups often intrinsically contain "dynamical" behaviour!
- Every infinite group G acts on the cantor space $\{0,1\}^G$ by homeomorphisms via the shift action via $L_g(\phi)(h) = \phi(g^{-1}h)$ for $\phi \in \{0,1\}^G$ and $g, h \in G$.
- The theory of group actions on manifolds is a rich area, yet many fundamental questions remain open.
- For instance, we do not know of a countable torsion-free group that does not admit a faithful action by orientation preserving homeomorphisms on the plane.
- The situation in dimension one is much nicer!

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One dimensional actions.

Groups of orientation preserving homeomorphisms of $\mathbb{R}$.

- G is *left orderable* if it admits a total order such that $\forall f, g, h \in G$, $f < g \implies hf < hg$.
- Left orderable groups are torsion-free (if $f > 1$, then $f^n > f^{n-1} > \ldots > f > 1$).
- (Fact:) A countable group G embeds into $\text{Homeo}^+(\mathbb{R})$ if and only if it is left orderable.
- Examples: free and surface groups (more generally, locally indicable groups), braid groups, torsion-free nilpotent groups, and virtually the fundamental groups of closed hyperbolic 3-manifolds. Thompson's group F!

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The normal rank of a group

- The *rank* of G, or $r(G)$, is the smallest size of a generating set.
- For $X \subseteq G$, $\langle\langle X \rangle\rangle$ is the *normal closure* of X in G: the smallest normal subgroup of G containing X.
- The *normal rank* of G is the smallest $n \in \mathbb{N}$ for which $\exists X \subset G, |X| = n$ such that $\langle\langle X \rangle\rangle = G$.

  Denote this by $n(G)$.

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- If G is abelian, $n(G) = r(G)$.
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The 1976 Wiegold problem on perfect groups

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Many of our favorite finitely generated perfect groups have normal rank one!

1. Higman's perfect group: $\langle f_0, \ldots, f_3 \mid f_i^{-1} f_{i+1} f_i = f_{i+1}^2 \ i \bmod 4 \rangle$.
2. $\mathrm{MCG}(S_g), g \geq 3$.
3. $\mathrm{PSL}_n(\mathbf{Z}), n \geq 3$.
4. $T \star T$ where T is Thompson's finitely presented infinite simple group.

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Finite perfect groups

Fact: (Wiegold) Finite perfect groups have normal rank one.

Proof: Perform an induction on $|G|$. If G is simple then we are done. Otherwise, let $N \trianglelefteq G$ be a non-trivial minimal normal subgroup. By our hypothesis, there is a $k' \in G/N$ such that $\langle\langle k' \rangle\rangle = G/N$. Let k be a lift of k' in G. It follows that $\langle\langle k \rangle\rangle \cdot N = G$.

Two cases:

1. $\langle\langle k \rangle\rangle \cap N = \{id\}$. So $G = \langle\langle k \rangle\rangle \oplus N$. Conclude by observing that if A, B are finite perfect groups such that $n(A) = 1 = n(B)$ then $n(A \oplus B) = 1$.
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Proof: Perform an induction on $|G|$. If G is simple then we are done. Otherwise, let $N \trianglelefteq G$ be a non-trivial minimal normal subgroup. By our hypothesis, there is a $k' \in G/N$ such that $\langle\langle k' \rangle\rangle = G/N$. Let k be a lift of k' in G. It follows that $\langle\langle k \rangle\rangle \cdot N = G$.
Two cases:

1. $\langle\langle k \rangle\rangle \cap N = \{id\}$. So $G = \langle\langle k \rangle\rangle \oplus N$. Conclude by observing that if A, B are finite perfect groups such that $n(A) = 1 = n(B)$ then $n(A \oplus B) = 1$.
2. $\langle\langle k \rangle\rangle \cap N \neq \{id\}$. By minimality of N, $N \leq \langle\langle k \rangle\rangle$ and so $\langle\langle k \rangle\rangle = G$.

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For a long time, it was suspected that the following should hold:

Let G, H be torsion-free finitely generated perfect groups. Then it should be the case that $n(G * H) > 1$.

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A solution to the Wiegold problem on perfect groups.

Theorem

(Chen, L. 2025) Let G, H be finitely generated and perfect left orderable groups. Then $n(G \star H) > 1$, solving the Wiegold problem on perfect groups.

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(Chen, L. 2025) Let G, H be countable left orderable groups. For any $w \in G \star H$ not conjugate into G, the natural map $G \hookrightarrow G \star H$ induces an injection $G \hookrightarrow (G \star H)/\langle\langle w \rangle\rangle$.

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A plethora of Finitely generated perfect left orderable groups exist:

- B'_n where B_n is the braid group and $n \geq 5$.
- The lift of Thompson's group T to $\mathbf{R}$.
- Higman's group.
- The lift of the $(2, 3, 7)$ triangle group to $\mathbf{R}$:

$$\langle f, g, h \mid f^2 = g^3 = h^7 = fgh \rangle$$

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Relationship to other conjectures

- (The Kervaire-Laudenbach conjecture) Let A be a nontrivial group. Consider the natural map $p : A \star \mathbf{Z} \to \mathbf{Z}$. The $A \hookrightarrow A \star \mathbf{Z}$ induces an inclusion $A \hookrightarrow A \star \mathbf{Z}/\langle\langle w \rangle\rangle$ whenever $p(w) \neq 0$.
  Theorem (Klyachko 1993) The conjecture holds when A is torsion-free.
  Theorem (Pestov 2008) The conjecture holds when A is hyperlinear.
- (The Levin conjecture) $n(A \star B) > 1$ whenever A, B are torsion-free.
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Relationship to 3-manifold topology

- (The Lickorish-Wallace theorem 1962) Any closed, connected, orientable 3-manifold can be obtained by performing a Dehn surgery on some n-component link in S^3.
- The minimal such n is called the *Dehn surgery number of the manifold*.
- Fact: The Dehn surgery number of M is bounded below by $n(\pi_1(M))$.

A natural question: Which manifolds can be described via the simplest possible surgery description, which is by surgery on a knot in S^3?

Corollary

(Chen, L. 2025) If $M = M_1 \# M_2$ where M_i are closed, connected, orientable 3-manifolds with $\pi_1(M)$ nontrivial and left orderable, then M cannot be obtained by performing a Dehn surgery on a knot in S^3.

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The following Corollary follows from remarks of Dunfield and Lidman.

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There are infinitely many different hyperbolic integer homology 3-spheres M such that $\pi_1(M)$ has normal rank at least 2, and M cannot be obtained from a Dehn surgery on a knot in S^3, $S^2 \times S^1$, or any lens space.

- It follows that there exist infinitely many solutions to the Wiegold problem that are hyperbolic, linear groups (and also freely irreducible).
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- Let A, B be countable left orderable groups.
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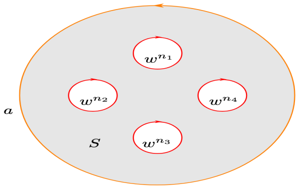
$$a = (g_1 w^{n_1} g_1^{-1}) \dots (g_k w^{n_k} g_k^{-1}) \qquad \text{for some } g_i \in A \star B, n_i \in \mathbf{Z} \setminus \{0\}$$

The structure of the proof.

- This gives rise to a surface map $f : \Sigma \to X$ where:



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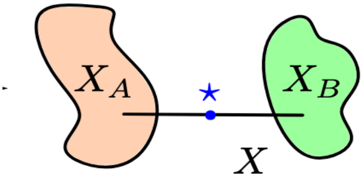

(b) The surface X. Here $\pi_1(X_A) = A, \pi_1(X_B) = B$

- Up to homotopy, we can assume that $f^{-1}(\star)$ is an embedded 1-manifold, and we cut the surface along this.
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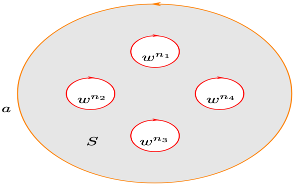

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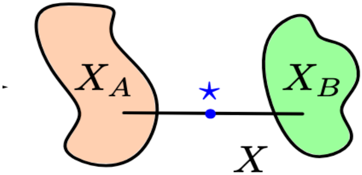

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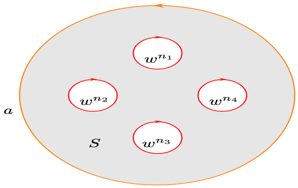

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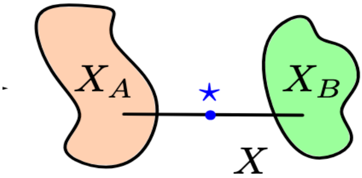

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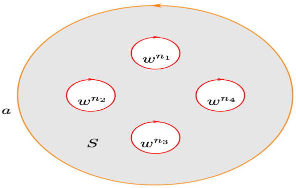

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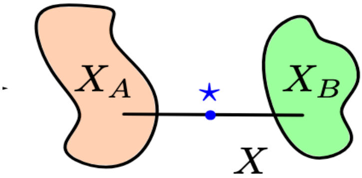

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The dynamical gadget: relative stackings.

- Consider an action $\sigma : A \star B \to \mathrm{Homeo}^+(\mathbf{R})$.
- For $x \in \mathbf{R}$, define the *w-trajectory*:

$$\Omega(w, x) = \{x \cdot \sigma(a_1 b_1 \dots a_i), x \cdot \sigma(a_1 b_1 \dots a_i b_i) \mid 1 \leq i \leq n\}$$

- $\Omega(w, x)$ is *stable* if $x \cdot \sigma(w) = x$ and $\Omega(w, x)$ does not contain any repeated elements.
- A relative stacking of w is a right action $\sigma : A \star B \to \mathrm{Homeo}^+(\mathbf{R})$ for which there exists $x \in \mathbf{R}$ such that $\Omega(w, x)$ is stable.

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The first reduction.

- Let H be a countable left orderable group.
- Given a fixed variable y, we shall consider *systems of equations and inequations over H*.
- Each equation in such a system will be of the form $y \cdot \alpha = y$ and each inequation will be of the form $y \cdot \beta \neq y$ where $\alpha, \beta \in H$ are fixed group elements.
- A solution to the system will be a pair (τ, x) where $\tau : H \to \text{Homeo}^+(\mathbf{R})$ is an action (not necessarily faithful) and $x \in \mathbf{R}$, so that:
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- Such a system is called a *finite system* if there are finitely many such equations and inequations.
- Given a system Λ, we denote by $\Lambda^{=}$ the set of equations in Λ and by $\Lambda^{\neq}$ the set of inequations in Λ.
- Given a finite collection of finite systems $\Lambda_1, \ldots, \Lambda_n$, we define a new system

$$\Xi(\Lambda_1, \ldots, \Lambda_n) := \Big(\bigcup_{1\leq i\leq n} \Lambda_i^{\neq}\Big) \cup \Big(\bigcap_{1\leq i\leq n} \Lambda_i^{=}\Big).$$

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For example, consider the Baumslag–Solitar group $\mathrm{BS}(1,2) = \langle f, g \mid f^{-1}gf = g^2 \rangle$, and consider the system $\Lambda = \{y \cdot f = y, y \cdot g \neq y\}$.

The action $\tau : \mathrm{BS}(1,2) \to \mathrm{Homeo}^+(\mathbf{R})$ given by:

$$t \cdot \tau(f) = 2t \qquad t \cdot \tau(g) = t + 1 \qquad \text{for } t \in \mathbf{R}$$

and the point $0 \in \mathbf{R}$ provides a solution. One checks that $0 \cdot \tau(f) = 0$ and $0 \cdot \tau(g) = 1 \neq 0$.

Lemma

Given a countable group H, let $\Lambda_1, \ldots, \Lambda_m$ be finite systems over H. If each Λ_i is solvable over H, then $\Xi(\Lambda_1, \ldots, \Lambda_m)$ is solvable over H.

We prove this using an inductive blow-up construction.

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A combinatorial challenge.

We reduce the problem to the following.

Lemma

Let A, B be countable right-orderable groups and $G = A \star B$. Consider a cyclically reduced word

$$w = a_1 b_1 \dots a_n b_n \in A \star B, \qquad a_i \in A \setminus \{id\}, b_i \in B \setminus \{id\},$$

and w_1 a proper prefix of w. Assume that w is not a proper power. Then there exists an action $\tau : A \star B \to \mathrm{Homeo}^+(\mathbf{R})$ and some $x \in \mathbf{R}$ such that $x \cdot \tau(w) = x$ and $x \cdot \tau(w_1) \neq x$.

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A combinatorial challenge: Method of proof.

To prove this, we build an action $\tau : A \star B \to \text{Homeo}^+(\mathbf{R})$ with a nonempty interval $I \subset \mathbf{R}$ such that:

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To construct such an action, we develop a method that we call *dynamical arrangements*, which are careful combinatorial encodings of actions of $A \star B$ on $\mathbf{R}$.

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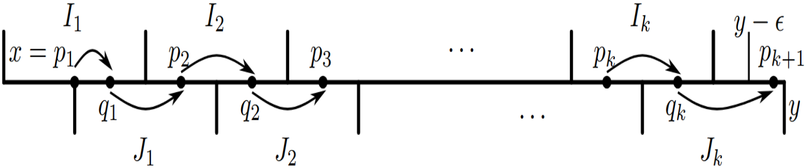

Figure: A catenation.

A combinatorial challenge: Method of proof.

Dynamical arrangements are built out of *Catenations*, which are described by pictures like these:

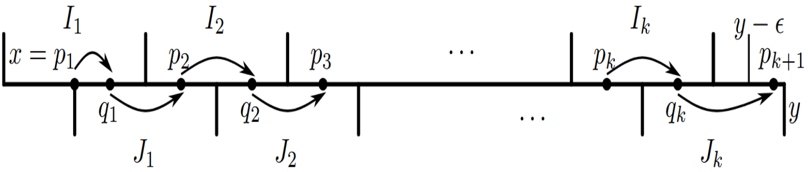

Figure: A catenation.

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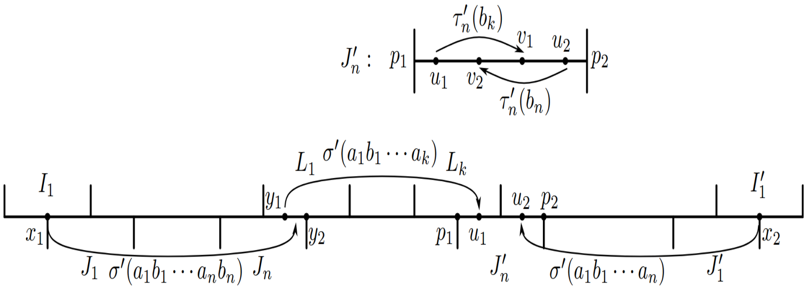

Figure: A dynamical arrangement.