Real Heegaard Floer homology

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Outline of the talk

1. Motivation: Real Seiberg-Witten theory, Miyazawa's exotic $\mathbb{RP}^2 \hookrightarrow S^4$;

2. Definition of $\widehat{HFR}(Y, \tau)$ for a 3-manifold Y with an involution τ;

3. Apply this to double branched covers of knots in S^3; take Euler characteristic $\rightsquigarrow$ new knot invariant $\chi_{\mathfrak{s}}(K)$. Is it related to anything we know?

Motivation from gauge theory

The Seiberg-Witten equations (1994) have led to:

- invariants of 4-manifolds that can detect exotic smooth structures: $X^4, \mathfrak{s} \in \mathrm{Spin}^c(X) \rightsquigarrow \mathrm{SW}(X, \mathfrak{s}) \in \mathbb{Z}$;
- monopole Floer homology, an invariant of 3-manifolds: $Y^3, \mathfrak{s} \in \mathrm{Spin}^c(X) \rightsquigarrow HM(Y, \mathfrak{s})$ (cf. **Kronheimer-Mrowka, M.**)

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Real Seiberg-Witten theory applies to manifolds with an involution τ with codimension 2 fixed point set.

The involution τ can be lifted to the spinor bundle in two ways (complex linearly and complex anti-linearly). The "real" theory corresponds to the complex anti-linear version, and to looking at solutions of the equations that are fixed by the involution.

Real Seiberg-Witten theory

In dimension 4, we get invariants $SWR(X, \tau, \mathfrak{s})$ where $s \in \mathrm{RSpin}^c(X, \tau)$ is a "real Spin^c structure" (**Tian-Wang** 2009; **Nakamura** 2010; **Kato** 2022; **Miyazawa** 2023; **Baraglia** 2025).

In particular, given $Z^2 \subset S^4$ smoothly embedded, we can construct the double branched cover $\Sigma_2(Z) \to S^4$ with branch locus Z, with the involution τ being the deck transformation.

$$|\deg(Z, \mathfrak{s})| = SWR(\Sigma_2(K), \tau, \mathfrak{s})$$

is called the Miyazawa invariant of the embedded surface Z.

In some cases (e.g. if $\Sigma_2(Z)$ is a homology S^4), there is a unique real Spin^c structure, and we just write $|\deg(Z)|$.

Real Seiberg-Witten theory

In dimension 3, we get real monopole Floer homology, $HMR(Y, \tau, \mathfrak{s})$, for $s \in \mathrm{RSpin}^c(Y, \tau)$ (**Konno-Miyazawa-Taniguchi** 2021; **Li** 2022).

In particular, given a knot $K \subset S^3$, we can form the double branched cover $\Sigma_2(K) \to S^3$ with branch locus K, with the involution τ being the deck transformation. Then $HMR(\Sigma_2(K), \tau, \mathfrak{s})$ is a knot invariant. Its Euler characteristic

$$|\deg(K, \mathfrak{s})| = \chi(HMR(\Sigma_2(K), \tau, \mathfrak{s}))$$

is called the Miyazawa invariant of the knot K.

In this case, the real Spin^c structures are in 1-1 correspondence with the Spin^c structures, and their number is the knot determinant

$$\det(K) = |H_1(\Sigma_2(K); \mathbb{Z})|.$$

If $\det(K) = 1$, we just write $|\deg(K)|$.

Applications of real SW theory

Two smoothly embedded surfaces $Z, Z' \subset S^4$ are called exotic if they are topologically but not smoothly isotopic.

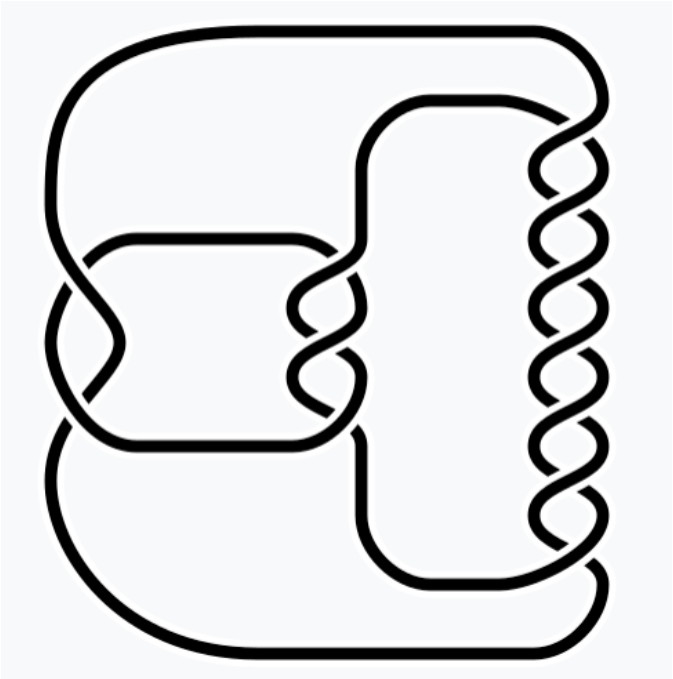
No orientable examples are known. (In particular, a famous open problem is the existence of exotic 2-knots, i.e. where Z and Z' are spheres.)

The first non-orientable examples were $(\mathbb{RP}^2)^{\#10}$ (**Finashin-Kreck-Viro**, 1988). Later, we had exotic $(\mathbb{RP}^2)^{\#n}$ for all $n \geq 5$, by work of **Finashin, Havens, Levine-Lidman-Piccirillo, Matić-Öztürk-Reyes-Stipsicz-Urzúa**. These are detected by the invariants of their double branched covers.

In 2023, **Miyazawa** used real SW theory to find (infinitely many) exotic $\mathbb{RP}^2$'s, and in fact exotic $(\mathbb{RP}^2)^{\#n}$ for all $n \geq 1$.

Miyazawa's example

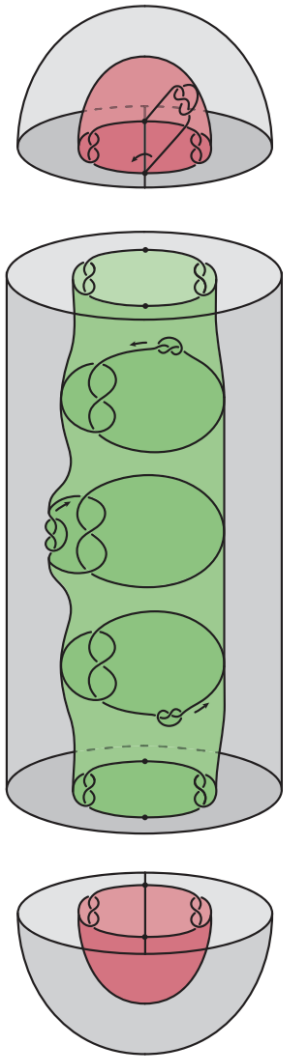
Consider the pretzel knot $P(-2,3,7) = 12n_{242}$, also known as the Fintushel-Stern knot:



It has determinant 1, and its double branched cover is the Brieskorn sphere $\Sigma(2,3,7)$. It is hyperbolic and has 7 exceptional slopes.

Roll spinning knots

Given any knot $K \subset S^3$, we can construct a 2-knot $\rho(K) \cong S^2 \subset S^4$, called the roll spun (figure by Naylor-Schwartz):



Miyazawa's example

Let $P_{\text{std}} = \mathbb{RP}^2 \subset \mathbb{CP}^2 \to \mathbb{CP}^2/\text{conjugation} = S^4$ be the standard embedding of $\mathbb{RP}^2$ in S^4. For $K = P(-2, 3, 7)$, let

$$P = P_{\text{std}} \# \rho(K) \subset S^4.$$

One can compute $\pi_1(S^4 - P_{\text{std}}) = \pi_1(S^4 - P) = \mathbb{Z}/2$, which implies (by a result of **Conway-Orson-Powell**) that P and P_{std} are topologically istotopic. However, they are not smoothly isotopic:

$$|\deg(P)| = |\deg(\rho(K))| = |\deg(K)| = 3,$$

whereas

$$|\deg(P_{\text{std}})| = |\deg(\rho(U))| = |\deg(U)| = 1,$$

where U is the unknot.

This gives an exotic $\mathbb{RP}^2$ in S^4.

More on Miyazawa's example

1. The double branched cover of $P = P_{\text{std}} \# \rho(K)$ was shown by **Hughes-Kim-Miller** to be the standard $\mathbb{CP}^2$, so this gives an exotic involution on $\mathbb{CP}^2$.

2. The double branched cover of $\rho(K)$ itself is a (potentially exotic) homotopy S^4.

3. **Miyazawa** obtained more exotic $\mathbb{RP}^2$ by considering $P_{\text{std}} \# \rho(K)^{\# n}$. **Kuhrman** showed that one can obtain even more examples using the Montesinos knots $K(2, 3, 6s \pm 1)$.

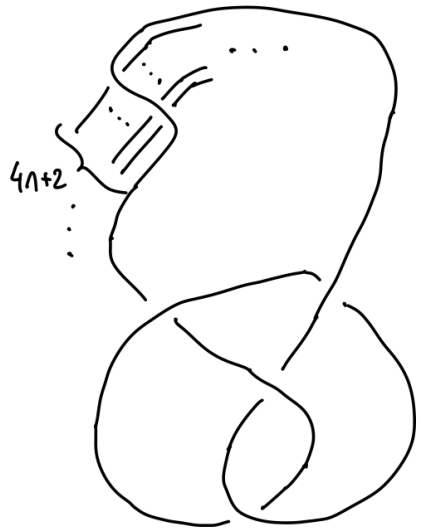
4. Miyazawa's invariant might also detect exotic 2-knots ($S^2 \hookrightarrow S^4$). Note that $\rho(K)$ itself doesn't work, because

$$\pi_1(S^4 - \rho(K)) \neq \pi_1(S^4 - S^2_{\text{std}}) = \mathbb{Z},$$

so $\rho(K)$ is not even topologically isotopic to S^2_{std}. One needs a better example.

Another application of real Seiberg-Witten theory

In 2024, **Kang, Park** and **Taniguchi** showed that the $(4n+2, 1)$-cables of the figure eight knot are not smoothly slice, generalizing the $n = 0$ result of **Dai-Kang-Mallick-Park-Stoffregen** (2022).



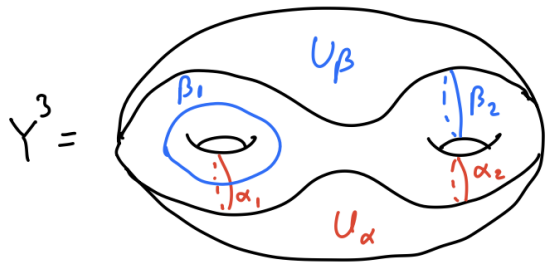
They also computed $|\deg(K)|$ for Montesinos knots (whose double branched covers are Seifert fibered).

Our motivation is to be able to compute it for *all* knots.

Heegaard Floer homology

In the early 2000's, **Ozsváth** and **Szabó** developed Heegaard Floer homology as a more computable alternative to Seiberg-Witten theory.

They start with a Heegaard decomposition of a 3-manifold Y:



then take the Lagrangian Floer homology of the tori

$$\mathbb{T}_\alpha = \alpha_1 \times \dots \alpha_g, \ \mathbb{T}_\beta = \beta_1 \times \cdots \times \beta_g \subset \mathrm{Sym}^g(\Sigma) = \Sigma^{\times g}/S_g.$$

There are different versions. We will focus on $\widehat{HF}(Y, \mathfrak{s})$, which counts holomorphic disks between $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ that avoid a basepoint $z \in \Sigma$. Here $\mathfrak{s}$ is a Spin^c structure.

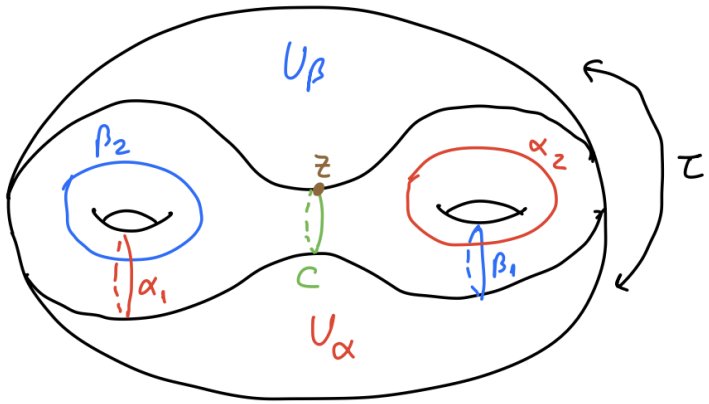
Real Heegaard Floer homology

Suppose Y^3 comes equipped with an involution τ such that its fixed point set is a nonempty link $C \subset Y$.

Theorem (Nagase 1973)

There exists a real Heegaard splitting, i.e. one such that $\tau(U_\alpha) = U_\beta$; then τ acts on $\Sigma \supset C$.

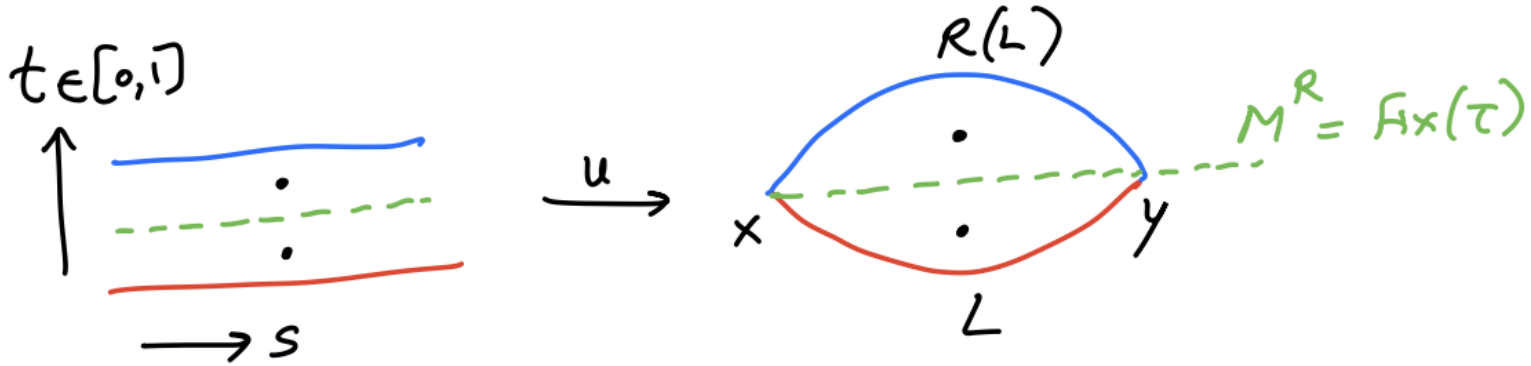
Hence, we can find a real Heegaard diagram, i.e. one with $\tau(\alpha_i) = \beta_i$. Here is one for S^3 with C =unknot:



Real Heegaard Floer homology

Given a real Heegaard diagram, the involution τ on Σ induces one (call it R) on the symplectic manifold $\mathrm{Sym}^g(\Sigma)$, so that $R(\mathbb{T}_\alpha) = \mathbb{T}_\beta$.

More generally, given a Lagrangian $L \subset (M, \omega)$ and an involution τ on M with $R^*(\omega) = -\omega$, under certain conditions we can define a real Lagrangian Floer homology $HFR(L)$ by counting holomorphic disks u with $R(u(s,t)) = u(s, 1-t)$:



Note that u is determined by its lower half, so $HFR(L)$ is just $HF(L, M^R)$.

Real Heegaard Floer homology

In our setting, we define

$$HFR(Y,\tau) = HFR(\mathbb{T}_\alpha) = HF(\mathbb{T}_\alpha, M^R)$$

but M^R is rather complicated, so it is better to think of symmetric holomorphic disks between $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$.

We have

$$HFR(Y,\tau) = \bigoplus_{\mathfrak{s}\in \mathrm{RSpin}^c(Y,\tau)} HFR(Y,\tau,\mathfrak{s})$$

where $\mathrm{RSpin}^c(Y,\tau)$ can be affinely identified with

$$\ker(\Theta : H^2(Y;\mathbb{Z}) \to H^2(Y/\tau;\mathbb{Z})) \times \frac{H^1(Y)^{\tau^*}}{\mathrm{im}(1+\tau^*)}.$$

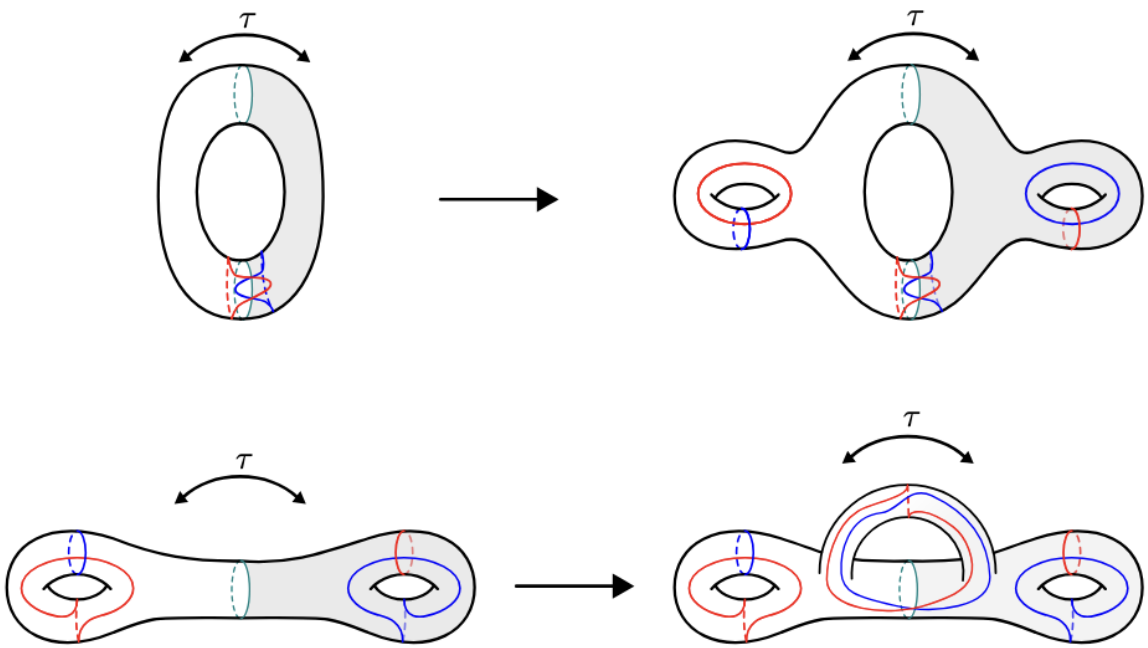
There are different versions of HFR: $-, +, \wedge, \infty$. We focus on $\wedge$ (in the complement of a basepoint z).

Invariance

Theorem (Guth-M.)

$HFR(Y, \tau, z, \mathfrak{s})$ is an invariant of $(Y, \tau, z, \mathfrak{s})$, up to isomorphism.

The proof involves checking invariance under real Heegaard moves: real isotopies and real handleslides (where the alpha and beta curves move in tandem), as well as two kinds of stabilizations:



More to do

We are currently working on proving naturality: that as we go between two Heegaard diagrams along two different sequences of real Heegaard moves, the induced maps on HFR are the same. This will prove that HFR is a canonical invariant of $(Y, \tau, z, \mathfrak{s})$.

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In the future we hope to also prove functoriality under four-dimensional cobordisms equipped with suitable involutions. This will lead to invariants of surfaces in 4-manifolds (in particular, surfaces in S^4), which should be more computable (and equal to those from real Seiberg-Witten theory, at least conjecturally).

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Until then, let's look at some 3-dimensional examples. We work over the ground field $\mathbb{F} = \mathbb{Z}/2$. (In some cases, this can be upgraded to $\mathbb{Z}$.)

Examples

When $Y = S^3$ (branched cover over S^3 with branch locus the unknot), we can use the trivial diagram of genus 0 with no α nor β curves, so $HFR = \mathbb{F}$.

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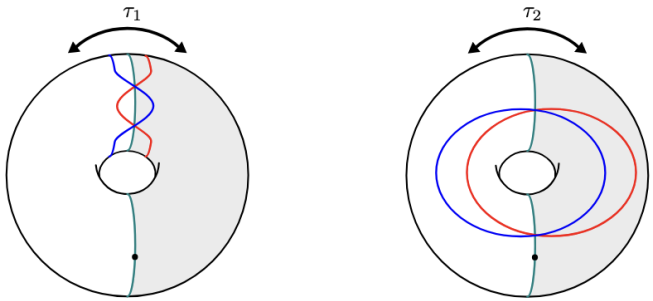
Here are two involutions on $S^1 \times S^2$:



First case: $(x, y) \to (-x, -y)$. We get $HFR = \mathbb{F} \oplus \mathbb{F}$, supported in degrees 0 and 1, in the torsion real Spin^c structure.

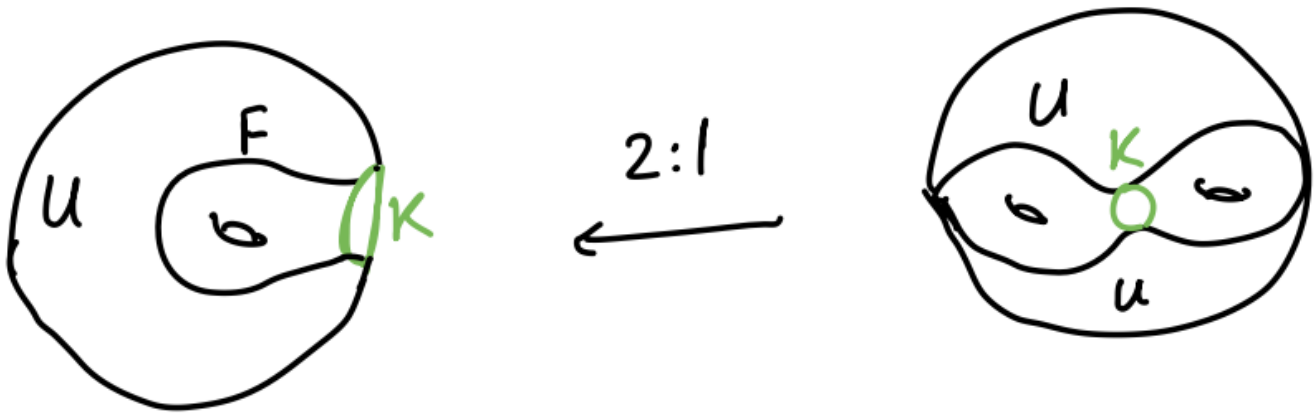
Second case: $(x, y) \to (x, \mathrm{rot}_\pi(y))$. We get $HFR = \mathbb{F} \oplus \mathbb{F}$, with the two summands in different real Spin^c structures.

Double branched covers

Suppose $K \subset S^3$ is a knot, and $(\Sigma_2(K), \tau)$ is the double branched cover.

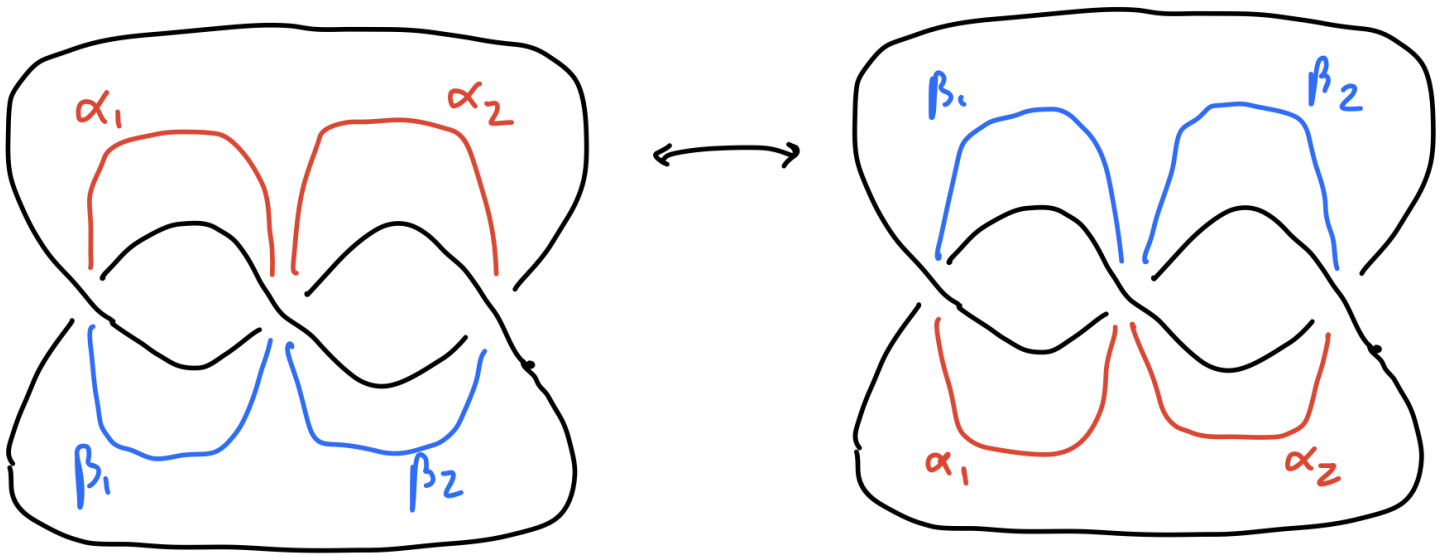
To construct a real Heegaard diagram, we can take any free spanning surface F for K; i.e. $F \subset S^3$ such that $\partial F = K$ and $S^3 - \text{nbhd}(F)$ is a handlebody. (Here, F can be orientable or not.)

Then the double of F gives the desired Heegaard surface:



Free spanning surfaces

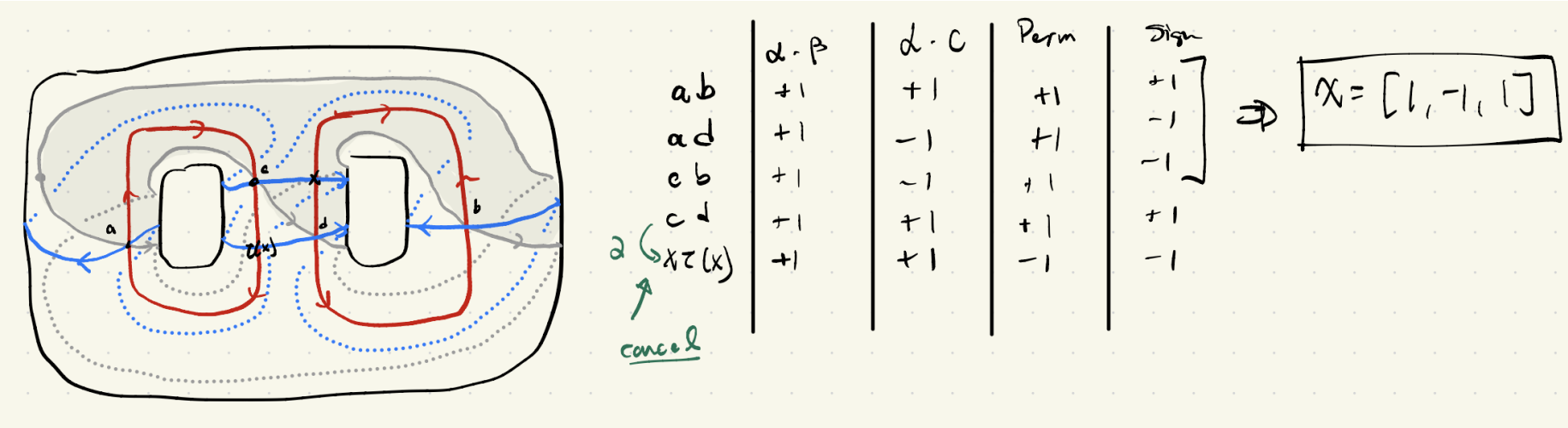
An example of a free spanning surface is the Seifert surface resulting from Seifert's algorithm applied to any knot diagram. Here is an example for the trefoil:



The generators of the real Floer complex are intersections $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ preserved by τ; i.e., intersections $\mathbb{T}_\alpha \cap M^R$.

Computations

Computing HFR is non-trivial (it involves counting invariant holomorphic disks), but it can be done for many small knots. Computing the Euler characteristic $\chi(HFR)$ is much easier, as it only involves counting the intersection points with signs (from the orientations of $\mathbb{T}_\alpha$ and M^R).



For the trefoil, we have $|\det(K)| = 3$ so there are three real Spin^c structures. We get $HFR = \mathbb{F}$ in each of them.

Computations

Theorem (Guth-M.)

The analogues of Miyazawa's invariant,

$$\chi_{\mathfrak{s}}(K) = \chi(HFR(\Sigma_2(K), \tau, \mathfrak{s})) \in \mathbb{Z}/\pm 1$$

and

$$|\deg(K)|_{\mathrm{HF}} = \chi(HFR(\Sigma_2(K), \tau)) = \sum_{\mathfrak{s}} \chi_{\mathfrak{s}}(K) = [\mathbb{T}_\alpha] \cdot [M^R]$$

are algorithmically computable for all knots $K \subset S^3$. (We fix the signs, for all $\mathfrak{s}$, so that $|\deg(K)|_{\mathrm{HF}} > 0$.)

In contrast, in real Seiberg-Witten theory the computations are limited (so far) to Montesinos and torus knots, whose branched double covers are Seifert fibered; and to (quasi-)alternating knots, when one can use the relation to ordinary $HM(\Sigma_2(K))$.

Calculations

Knot	$\det(K)$	$\lvert\deg(K)\rvert_{HF}$	$\chi_{\mathfrak{s}}(K)$
3_1	3	1	$[1,-1,1]$
4_1	5	3	$[1,1,-1,1,1]$
5_1	5	1	$[-1,1,1,1,-1]$
5_2	7	3	$[1,1,-1,1,-1,1,1]$
6_1	9	5	$[1,1,1,-1,1,-1,1,1,1]$
6_2	11	1	$[1,1,-1,1,-1,-1,-1,1,-1,1,1]$
6_3	13	3	$[1,-1,1,1,1,-1,-1,-1,1,1,1,-1,1]$
7_1	7	1	$[-1,1,1,-1,1,1,-1]$
7_2	11	5	$[1,-1,1,1,1,-1,1,1,1,-1,1]$
7_3	13	1	$[-1,1,1,1,-1,-1,1,-1,-1,1,1,1,-1]$
10_{152}	11	1	$[-1,-1,1,1,-1,3,-1,1,1,-1,-1]$
$12n_{121}$	1	3	$[3]$
$12n_{242}$	1	3	$[3]$

Calculations

We find that $\chi_s = \pm 1$ for all small knots (up to 8 crossings), but this stops being true for larger knots (e.g. 10_{152}). **Hendricks** (2025) showed that $\chi_s = \pm 1$ for all (quasi-)alternating knots.

For the pretzel knot $P(-2,3,7) = 12n_{242}$ used by Miyazawa to find an exotic $\mathbb{RP}^2 \subset S^4$, we get $\chi_{\mathfrak{s}} = 3$, in agreement with his calculation.

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In other Floer theories, the Euler characteristic is a more classical invariant; e.g. the Casson invariant for instanton Floer homology, Turaev torsion for monopole or Heegaard Floer homology, the Alexander polynomial for knot Floer homology, the knot determinant for HF of double branched covers, the knot signature for a version of knot instanton homology.

Open question: Are $\chi_{\mathfrak{s}}(K)$ and $|\deg(K)|_{\mathrm{HF}}$ related to better known knot invariants?