Seifert surfaces for alternating knots (in 4D)

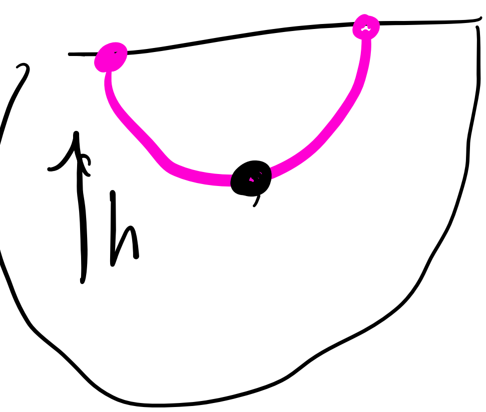
with Seungwon Kim
and Jaehoon Yoo

Motivating question: when are two Seifert surfaces isotopic in 4D?

Obsesvation

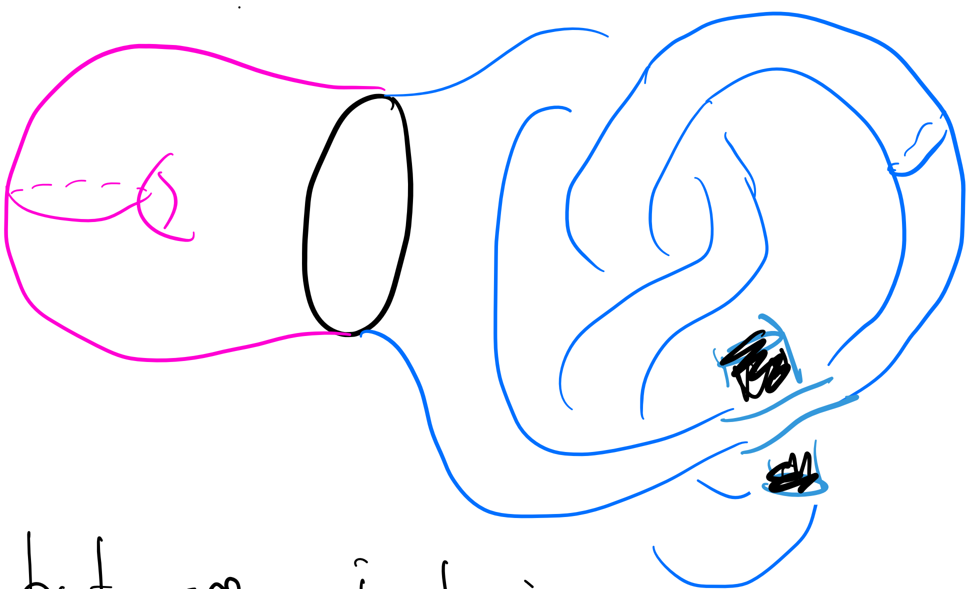
once we push interior Seifert F into B^4

$\pi_1(B^4 \setminus F) \cong \mathbb{Z}$



Fact: handle decomp of comp B^4 $\longleftrightarrow$ crit pts in $h|_F$

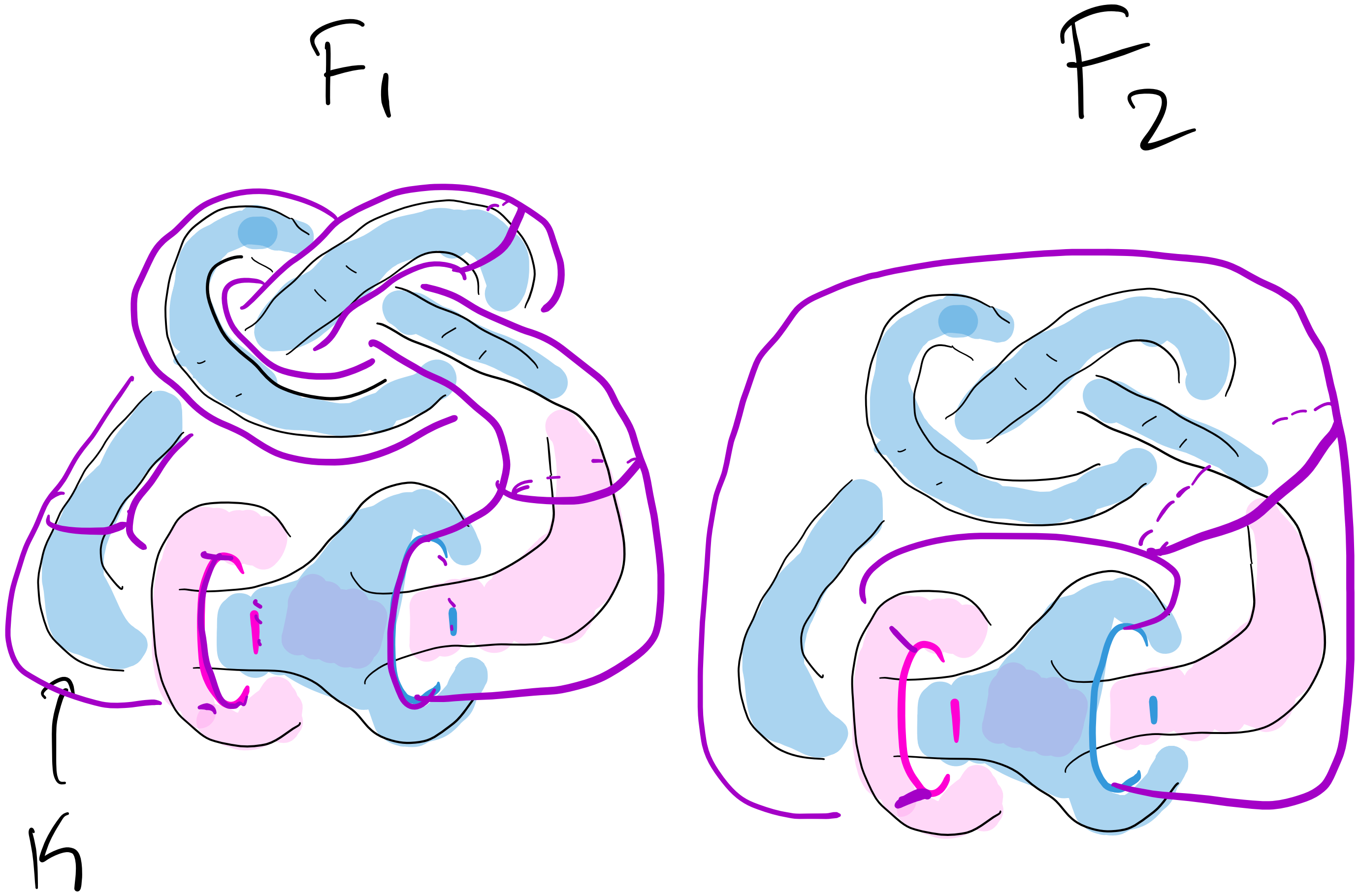
not isotopic in S^3



but are isotopic rel ∂ if we push their interiors into B^4 ($\partial B^4 = S^3$)

This example is compressible –

is there a more interesting example?



Answer: yes! But these are still iso rel ∂ when their interiors are pushed into B^4.

(Example: Alford 1970)

Thm Livingston 1982

Every two connected, genus-g Seifert surface for k-comp unlink are iso rel ∂ in B^4.

"Surely not the case for every boundary."

In literature

Hayden-Kim-M-Park-Sundberg checked many different sources constructing Seifert surfaces.

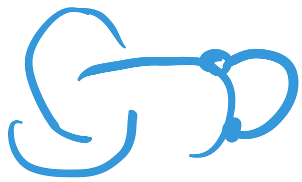
Most "obviously" isotopic in B^4; see Section 2.

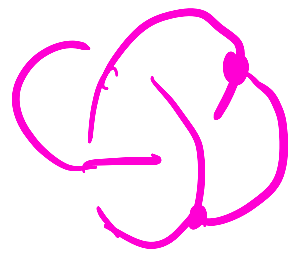
But in general, genus-g Seifert surfaces for K need not be isotopic in B^4.

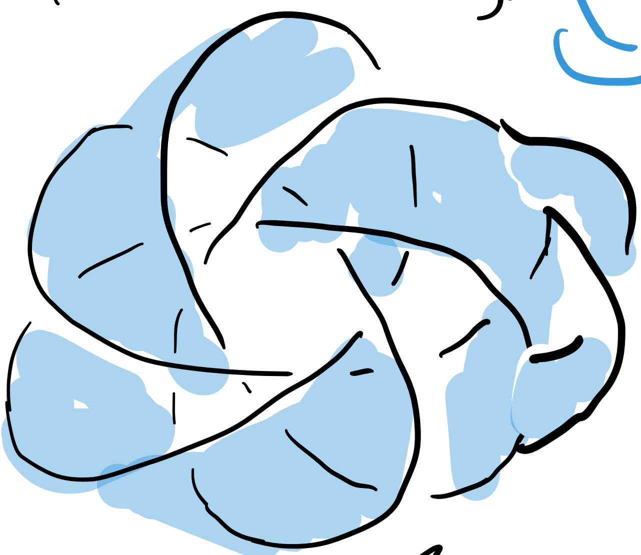
$\pi_1 = F^2$

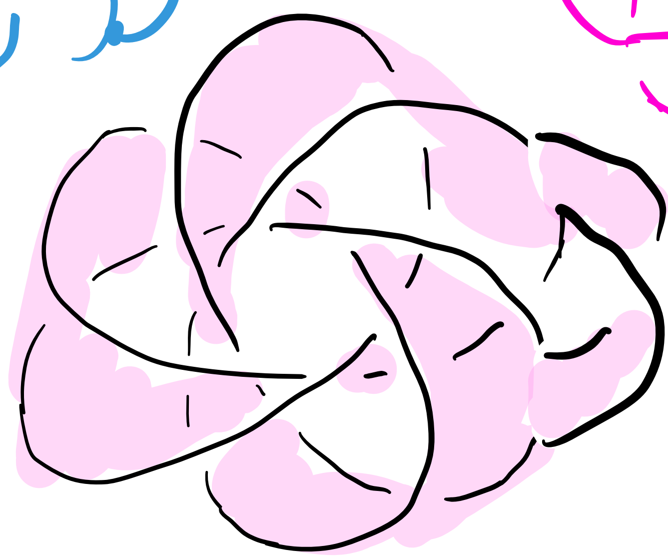
Hayden-Kim-M-Park-Sundberg 2022

$\pi_1 = \pi_1(S^3 \setminus \text{tref}) * \mathbb{Z}$









not iso in S^3

(And now even more examples in Aka-Feller-A.B. Miller-Wieser)

Still open: can a knot band infinitely many distinct genus-g Seifert surfaces up to isotopy in B^4?

(We construct links with ∞-family distinct surfaces

- connected Seifert

- 3-comp unlink bands ∞-family of distinct $(D \cup A)$)

Given a knot K, can we count min genus Seifert surfaces up to isotopy in B^4?

<u>Question</u>

How many min-genus Seifert surfaces does a hyperbolic knot K bound in B^4?

(Can be more than 1, but of course is $< \infty$.)

In general: how many Seifert surfaces in B^4?

Thm (Kim-M-Yoo)

Any two genus-g Seifert surfaces for a non-split alternating link L are isotopic in B^4.

<u>Rmk</u>

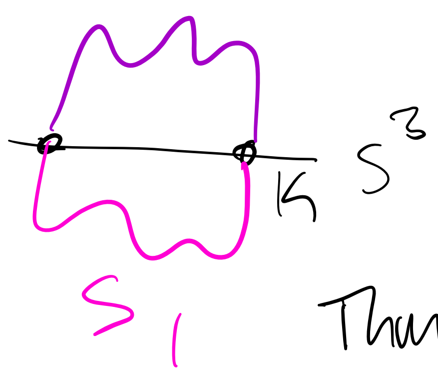
Every Seifert surface for non-split alternating link L is connected

[Lickorish Prop 4.8
Adams-Kindred 2013 Cor 5.2]

so no distinction between genus, Euler characteristic.

Open question (*)

Given a knot K and two Seifert surfaces S_1, S_2 for K, is $S_1 \cup \bar{S}_2 \subset S_4$ smoothly unknotted?



"unknotted" means bounds a genus-g solid.

Thm (Conway - Powell 2020)
Yes topologically.

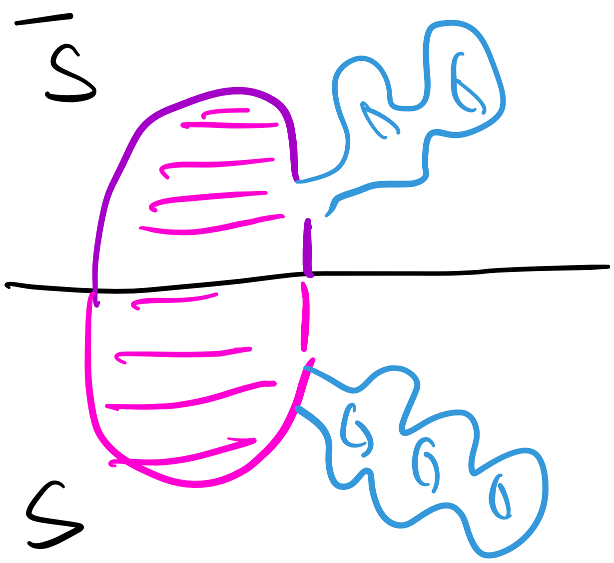
Consequence of main thm

Answer to (*) is "yes" for alternating knots.

Cor Answer to (*) is "yes" for K alternating.

Say min genus surface is S.

$S_1 = S \# n \text{ tori}$ $\quad S_2 = S \# m \text{ tori}$



$= \underbrace{(S \cup \bar{S})}_{\partial\, \underbrace{S \times I}_{\text{solid of genus } 2g}} \# \underbrace{(m+n)}_{\text{unknot}} \text{tori}$

Bigger question

Given an oriented surface in S^4 that is top unknotted, is it also smoothly unknotted?

$(*)$ is a special case.

Strategy of main thm:

Understand space of all

- Seifert surfaces up to genus g for L

- Prove theorem inductively

<u>Def</u> (Kakimizu 1992)

The Kakimizu complex $IS(L)$ for a non-split link L is a simplicial complex with

(isotopy rel ∂ classes)

0-cells $\longleftrightarrow$ incompressible Seifert surfaces for L

n-cells $\longleftrightarrow$ bounded by $n+1$ 0-cells when rep by disjoint (interior) Seif surfaces

contains subcomplex $\underbrace{MS(L)}_{\text{min genus}}$

Kakimizu showed in original paper that $MS(L)$ and $IS(L)$ are connected.

<u>Def</u> KMY

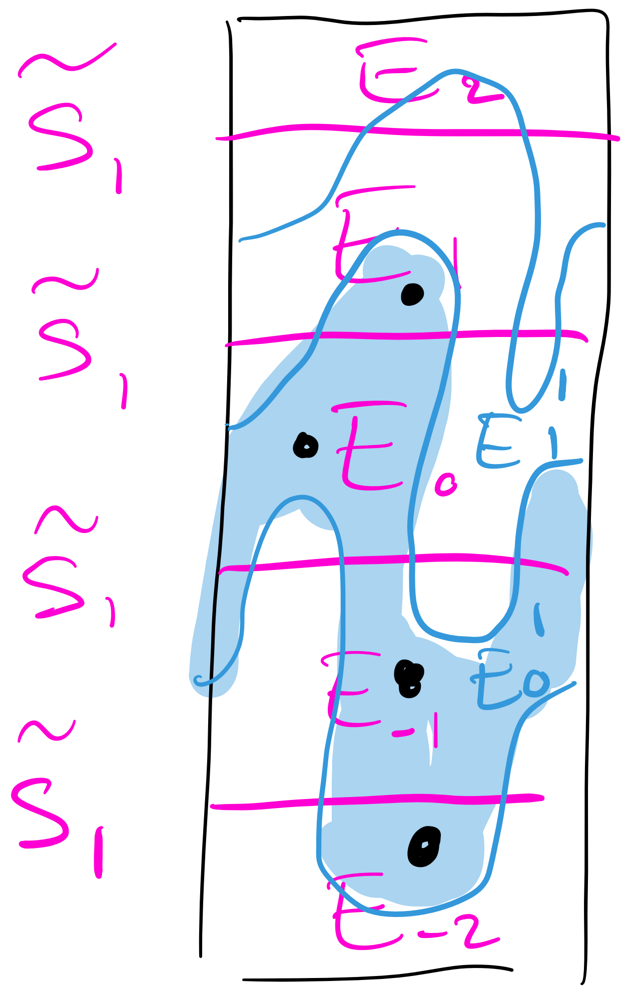
$IS_k(L)$ = subcomplex includes vertices rep by Σ $\chi(\Sigma) \geq k$

Thm (KMY but following Kakimizu) $IS_k(L)$ is connected for every k.

Pf

Kakimizu showed

infinite cyclic cover $S^3 \setminus L$



lifts of S_2

$d(S_1, S_2) =$ in $IS(L)$

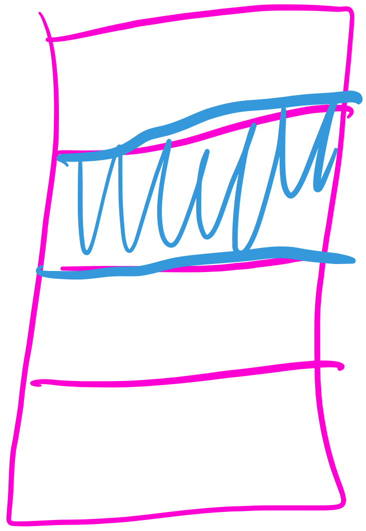
(minimum)

$\#$ of E_i s that E'_0 intersects -1

$d(S_1, S_2) = 0$

$\iff E_0'$ intersects one

E_i — so



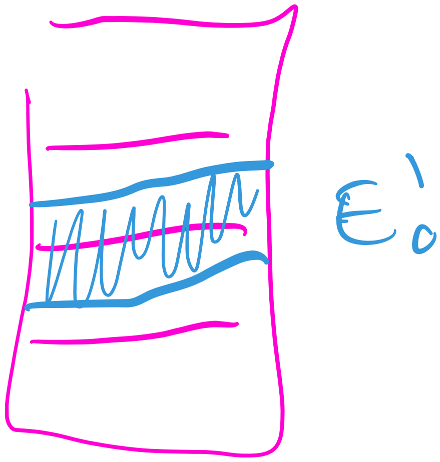
S_1, S_2 iso

rel ∂

$d(S_1, S_2) = 1$

$\iff E_0'$ intersect two

E_i —



S_1, S_2 disjoint

interior.

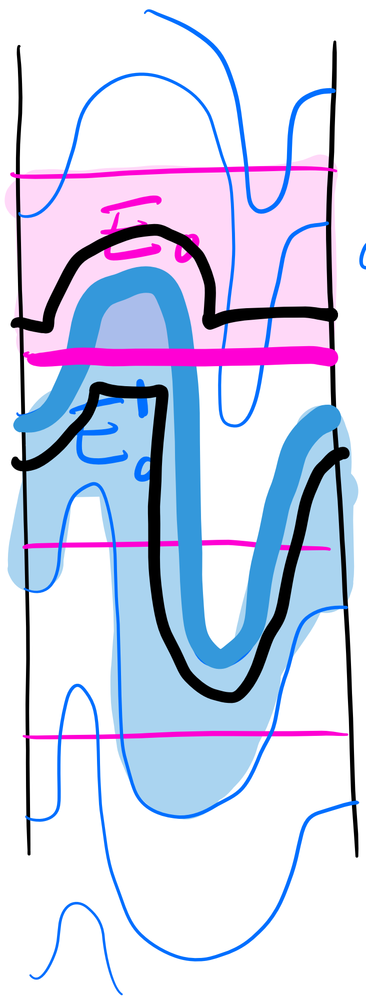
Key Lemma

(Kakimizu) Let S_1, S_2 be incompressible Seifert surfaces for L with $d(S_1, S_2) = n > 0$. Then there exists another incompressible Seifert surface Σ for L s.t. $\mathring{\Sigma} \cap \mathring{S}_2 = \emptyset$ $(d(S_1, S_2) = 1)$ and $d(\Sigma, S_1) < n$.

Moreover, $\chi(\Sigma) \geq \min\{\chi(S_1), \chi(S_2)\}$.

(connected- $g(\Sigma) \leq \max\{g(S_1), g(S_2)\}$)

Schematic



$$\chi(\Sigma)+\chi(\Sigma') = \chi(S_1)+\chi(S_2)$$

WLOG
$$\chi(\Sigma) \geq \min\{\chi(S_1), \chi(S_2)\}$$

$d=3$

Kakimizu obtains the surface in the infinite cyclic cover of link complement by cutting/pasting lifts of S_1, S_2, then projecting down. The cut/paste has two components - just take care to choose the one with less negative Euler characteristic!

Consequence:

There exist incompressible Seifert surfaces $\Sigma_1, \dots, \Sigma_k$ s.t. $S_1 = \Sigma_1$, $S_2 = \Sigma_k$, $\mathring{\Sigma}_i \cap \mathring{\Sigma}_{i+1} = \emptyset \quad \forall i$

and $\chi(\Sigma_i) \geq \min\{\chi(S_1), \chi(S_2)\}$.

Another key lemma about Seifert surfaces for alternating links

Thm (Kindred 2018)

Tube crossing lemma

If L non-split alt link

F closed surface

containing L, essential

then for any reduced, alternating diagram of L there is some crossing c so above c, F looks like ($**$) [upcoming.]

Convention:

Given a surface F containing a link L, we say F essential (with respect to L) if each component of $F \setminus \nu(L)$ is essential in $S^3 \setminus \nu(L)$.

(∗∗)

$F =$ 

$\leftarrow$ none of F going through this tube!

Summary so far:

Kakimizu complex $IS(L)$

- vertices $\leftrightarrow$ incompressible Seifert surfaces

- filtration $IS_k(L)$
  vertices $\leftrightarrow$ incompressible Seifert surfaces $\chi \geq k$

connected!

- Tube crossing lemma

Lemma

Let Σ_1, Σ_2 be genus-g Seifert surfaces for non-split alternating L. Let S_1, S_2 be incompressible Seifert surfaces obtained by compressing Σ_1, Σ_2. Then if

$d(S_1, S_2) \leq 1$ then

Σ_1, Σ_2 iso in B^4 (rel ∂)

Case 1: S_1, S_2 isotopic

(ie $d(S_1, S_2) = 0$)

Then each of Σ_1, Σ_2 obtained by attaching tubes to $S_1 = S_2$ in B^4 $S_1 \#$ kri $= S_2 \#$ kri

Case 2: $d(S_1, S_2) = 1$

Induct again on $\chi(\Sigma_i)$ (base is 1, decreasing)

$\chi(\Sigma_i) = 1$ then $\Sigma_i = D^2$ bounded by unknot

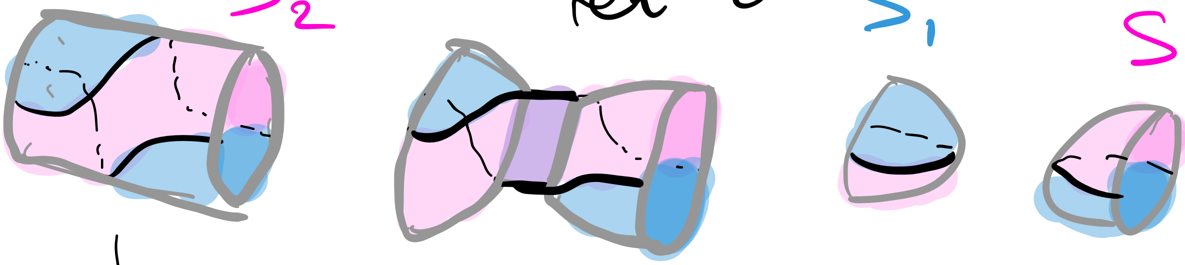
true

after attaching tubes

to make genus same, so are iso rel ∂

iso rel ∂ in B^4 (after correcting genus)

S_1 S_2 S_1' S_2'



tube crossing

diagram of L

reduced, alternating

L'

alternating, non-split

S_0 Σ_1, Σ_2 are iso rel ∂ in B^4.

Now whole theorem

Thm (Kim - M - Yoo)

If Σ_1, Σ_2 are genus-g Seifert surfaces for a non-split, alternating link, then Σ_1, Σ_2 are isotopic rel ∂ in B^4.

Pf.

genus-g

$$\begin{tikzcd}
\Sigma_1 \arrow[d, "\text{compress}"] & \Sigma' \arrow[d, "\text{compress}"] & \Sigma'' \arrow[d] & & \Sigma''' \arrow[d] & \Sigma_2 \arrow[d, "\text{compress}"] \\
S_1 \arrow[r] & F_1 \arrow[r] & F_2 \arrow[r] & \cdots \arrow[r] & F_n \arrow[r] & S_2
\end{tikzcd}$$

$IS_k(L)$

Σ_1, Σ_2 iso rel ∂ in B^4.

Cor. For L a non-split alternating link, L bounds a **unique** min genus Seifert surface S (up to iso rel ∂ in B^4!) and every other Seifert surface is isotopic rel ∂ in B^4 to

$$S \# (\text{unknotted tori})$$