Homogeneous braids are visually prime

Joint with Peter Feller and Lukas Lewark

Miguel Orbegozo Rodriguez

Université de Neuchâtel

20 February 2025

Primeness of links

Definition

A *decomposition sphere* for a link L is an embedded $S^2 \subset S^3$, intersecting L in two points, such that each component of $S^3 \setminus S^2$ contains a nontrivial link.

Such a sphere exhibits L as a connected sum $L = L_1 \# L_2$ (in particular L is non prime). If such a sphere does not exist then L is *prime*.

Diagrams

Links are most commonly represented using *diagrams*, which are projections to a plane.

Diagrams

Links are most commonly represented using *diagrams*, which are projections to a plane.

Definition

A *decomposition circle* for a diagram D is an embedded S^1 in the plane that intersects D in two points, and gives rise to a decomposition sphere of the associated link.

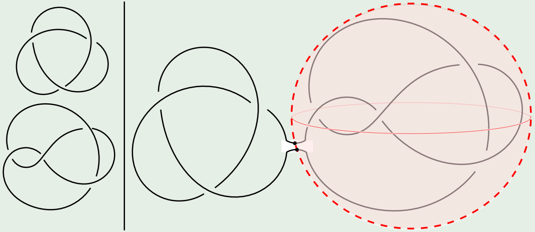
Diagrams

Links are most commonly represented using *diagrams*, which are projections to a plane.

Definition

A *decomposition circle* for a diagram D is an embedded S^1 in the plane that intersects D in two points, and gives rise to a decomposition sphere of the associated link.

Example



Primeness in diagrams

Diagrams are $2D$, links are $3D$. Do we lose information?

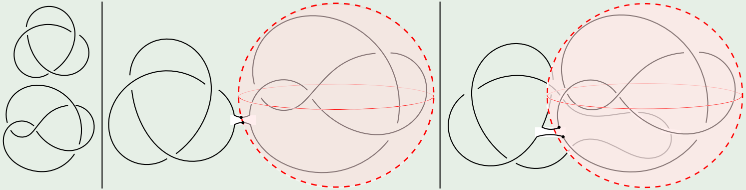
Primeness in diagrams

Diagrams are $2D$, links are $3D$. Do we lose information? For example, the boundary of the projection of a decomposition sphere might intersect the projection of a link more than twice, thus not giving a decomposition circle.

Primeness in diagrams

Diagrams are $2D$, links are $3D$. Do we lose information? For example, the boundary of the projection of a decomposition sphere might intersect the projection of a link more than twice, thus not giving a decomposition circle.

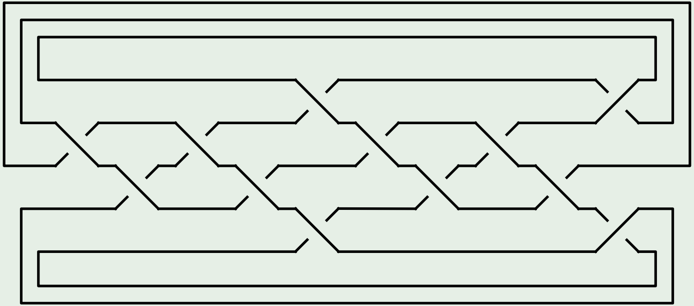
Example



Primeness in diagrams

Diagrams are $2D$, links are $3D$. Do we lose information? For example, the boundary of the projection of a decomposition sphere might intersect the projection of a link more than twice, thus not giving a decomposition circle.

Example (Morton)

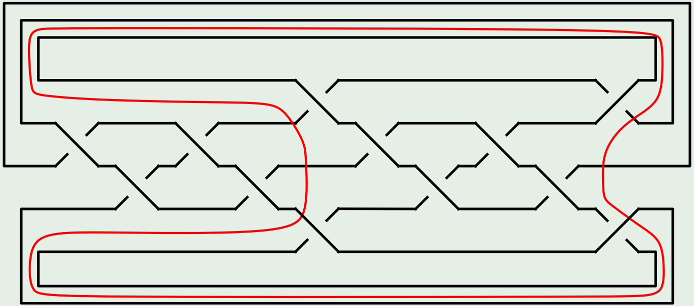


$$\sigma_2\sigma_3\sigma_2\sigma_3\sigma_1\sigma_4\sigma_2\sigma_3\sigma_2\sigma_3\sigma_1^{-1}\sigma_4^{-1}$$

Primeness in diagrams

Diagrams are $2D$, links are $3D$. Do we lose information? For example, the boundary of the projection of a decomposition sphere might intersect the projection of a link more than twice, thus not giving a decomposition circle.

Example (Morton)



$$\sigma_2\sigma_3\sigma_2\sigma_3\sigma_1\sigma_4\sigma_2\sigma_3\sigma_2\sigma_3\sigma_1^{-1}\sigma_4^{-1}$$

History of the problem

Question

In which classes of diagrams can we detect whether the associated links are prime?

History of the problem

Question

In which classes of diagrams can we detect whether the associated links are prime?

Clearly not in all classes of diagrams, and by Morton, not even all braid diagrams!

History of the problem

Question

In which classes of diagrams can we detect whether the associated links are prime?

Clearly not in all classes of diagrams, and by Morton, not even all braid diagrams!

(Partial) answers

- *Alternating diagrams (Menasco '84).*

History of the problem

Question

In which classes of diagrams can we detect whether the associated links are prime?

Clearly not in all classes of diagrams, and by Morton, not even all braid diagrams!

(Partial) answers

- *Alternating diagrams (Menasco '84).*
- *Positive (and negative) braids (Cromwell '88).*

History of the problem

Question

In which classes of diagrams can we detect whether the associated links are prime?

Clearly not in all classes of diagrams, and by Morton, not even all braid diagrams!

(Partial) answers

- *Alternating diagrams (Menasco '84).*
- *Positive (and negative) braids (Cromwell '88).*
- *Positive diagrams (Ozawa '02).*

History of the problem

Question

In which classes of diagrams can we detect whether the associated links are prime?

Clearly not in all classes of diagrams, and by Morton, not even all braid diagrams!

(Partial) answers

- *Alternating diagrams (Menasco '84).*
- *Positive (and negative) braids (Cromwell '88).*
- *Positive diagrams (Ozawa '02).*

Conjecture (Cromwell '88)

Diagrams where Seifert's algorithm yields a minimal genus surface.

Our aim in this project (in progress) is to prove this conjecture.

History of the problem

Remark (Cromwell '88)

For diagrams given by braids, minimal genus surface from Seifert's algorithm $\iff$ *braid is homogeneous.*

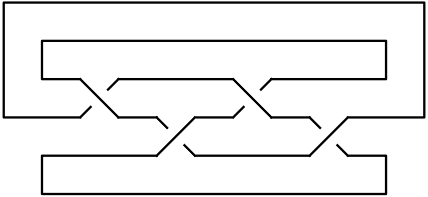
History of the problem

Remark (Cromwell '88)

For diagrams given by braids, minimal genus surface from Seifert's algorithm $\iff$ *braid is homogeneous.*

Definition

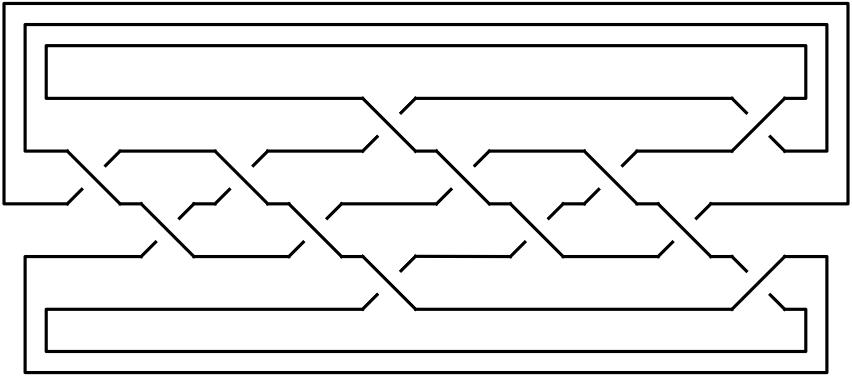
A braid word β is *homogeneous* if each standard generator σ_i appears with the same sign throughout.



$\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}$

Morton's example is not a homogeneous braid

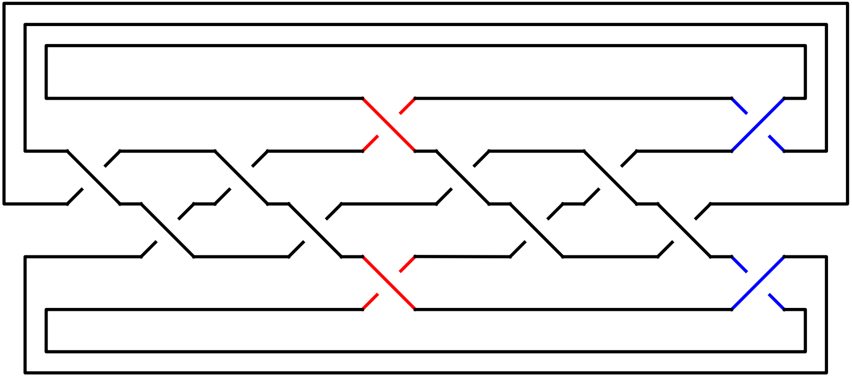
In Morton's example, both σ_1 and σ_4 appear with positive **and** negative sign.



$$\sigma_2\sigma_3\sigma_2\sigma_3\sigma_1\sigma_4\sigma_2\sigma_3\sigma_2\sigma_3\sigma_1^{-1}\sigma_4^{-1}$$

Morton's example is not a homogeneous braid

In Morton's example, both σ_1 and σ_4 appear with positive **and** negative sign.



$$\sigma_2\sigma_3\sigma_2\sigma_3\sigma_1\sigma_4\sigma_2\sigma_3\sigma_2\sigma_3\sigma_1^{-1}\sigma_4^{-1}$$

The result

The result

We confirm Cromwell's conjecture for braid diagrams.

Theorem (Feller-Lewark-O. '24)

If a homogeneous braid diagram does not have a decomposition circle, then its associated link is prime.

In Cromwell's language, *homogeneous braids are visually prime*.

Structure of homogeneous braids

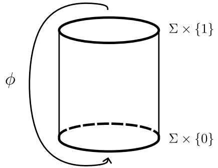
1. Homogeneous braids are fibered $\rightsquigarrow$ Open book techniques.

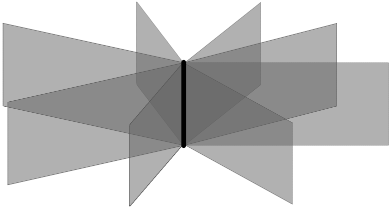
Open books

Definition

A link L is fibered if its complement can be written as $S^3 \setminus \nu L = \frac{\Sigma \times [0,1]}{(x,1) \sim (\phi(x),0)}$ for some Seifert surface Σ (the *page*) and some diffeomorphism ϕ which is the identity near $\partial\Sigma$ (the *monodromy*). The pair (Σ, ϕ) is an *open book*.

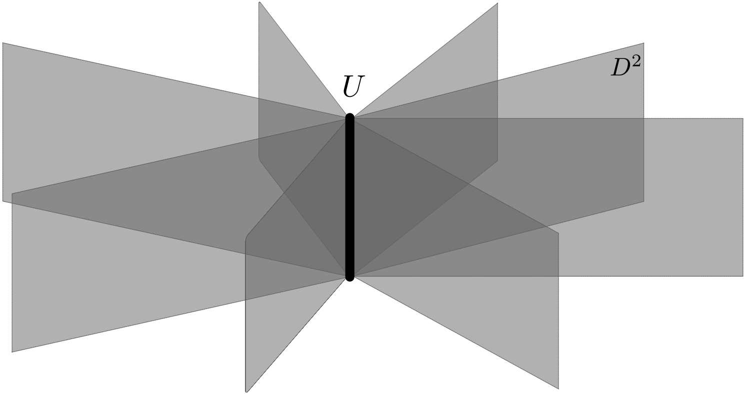
Remark. If a link is fibered, the surface Σ is unique up to isotopy.





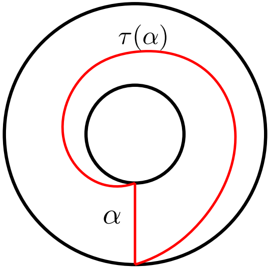
Open book examples

- The unknot is a fibered knot, with open book description (D^2, Id).



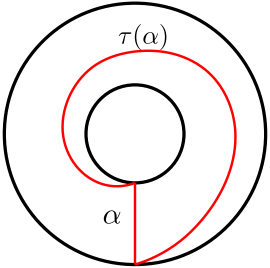
Open book examples

- The Hopf link is a fibered link, with open book description (A, τ).



Open book examples

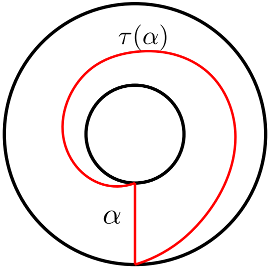
- The Hopf link is a fibered link, with open book description (A, τ).



Here, we have described the diffeomorphism τ by indicating the image of an arc.

Open book examples

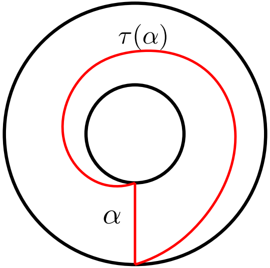
- The Hopf link is a fibered link, with open book description (A, τ).



Here, we have described the diffeomorphism τ by indicating the image of an arc. In general, if we have a basis of arcs, their images determine the diffeomorphism up to isotopy.

Open book examples

- The Hopf link is a fibered link, with open book description (A, τ).



Here, we have described the diffeomorphism τ by indicating the image of an arc. In general, if we have a basis of arcs, their images determine the diffeomorphism up to isotopy. **Aim:** Study open books (and thus the associated links) via their effect on arcs.

Arcs in open books

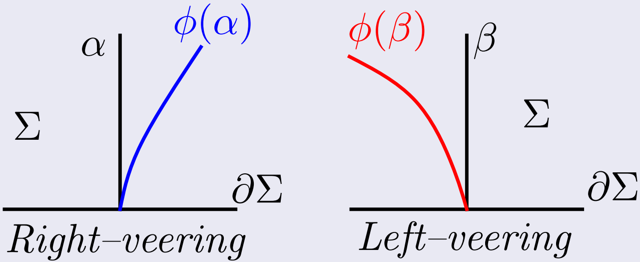
Definition

For an open book (Σ, ϕ), an oriented arc α such that α is isotopic to $\phi(\alpha)$ is *fixed*.

Arcs in open books

Definition

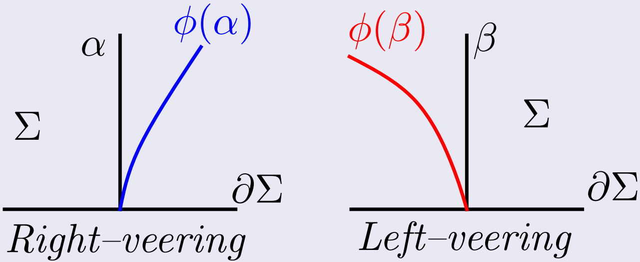
For an open book (Σ, ϕ), an oriented arc α such that α is isotopic to $\phi(\alpha)$ is *fixed*.Otherwise, it is either *right–veering* or *left–veering*.



Arcs in open books

Definition

For an open book (Σ, ϕ), an oriented arc α such that α is isotopic to $\phi(\alpha)$ is *fixed*.Otherwise, it is either *right–veering* or *left–veering*.

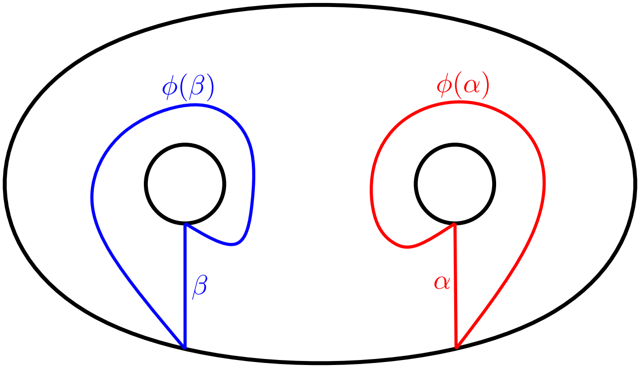


Remark

In a (prime) positive braid, every arc is right–veering.

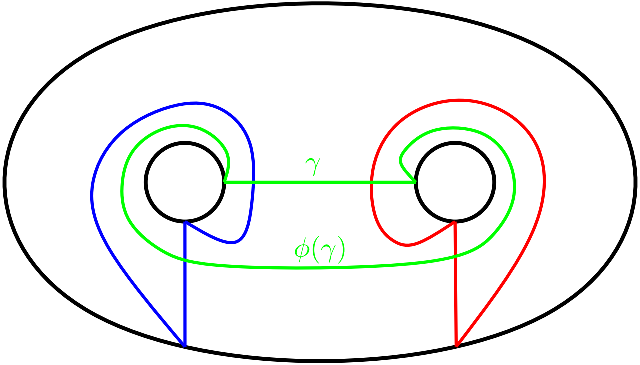
Arcs in open books

An example of a right-veering and a left-veering arc in an open book.



Arcs in open books

Orientation matters: the arc γ is right-veering or left-veering depending on the starting point.



Arcs in open books

Why are we interested in the images of arcs?

Arcs in open books

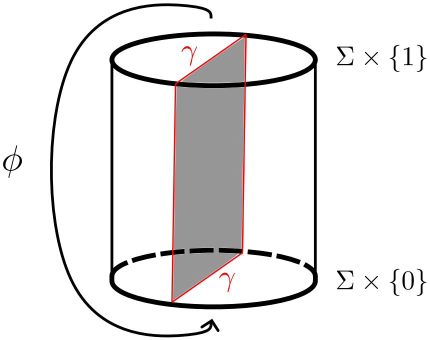
Why are we interested in the images of arcs?

Fact. A fibered link in S^3 is composite if and only if there exists a (non-boundary parallel) fixed arc in its fiber surface.

Arcs in open books

Why are we interested in the images of arcs?

Fact. A fibered link in S^3 is composite if and only if there exists a (non-boundary parallel) fixed arc in its fiber surface.

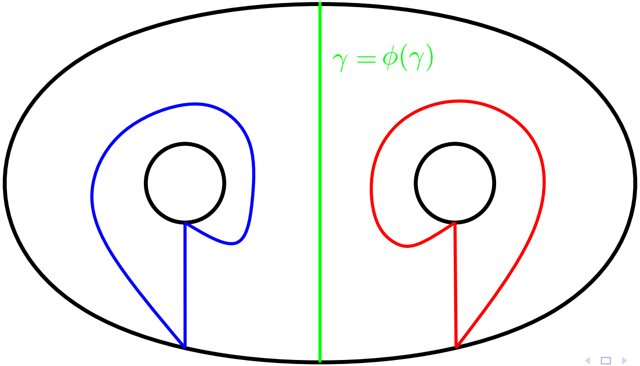


Arcs in open books

Why are we interested in the images of arcs?

Fact. A fibered link in S^3 is composite if and only if there exists a (non-boundary parallel) fixed arc in its fiber surface.

The arc γ is fixed, so the link is composite (it is the connected sum of a positive and a negative Hopf link).



Structure of homogeneous braids

1. Homogeneous braids are fibered $\rightsquigarrow$ Open books $\rightsquigarrow$ Fixed arcs.

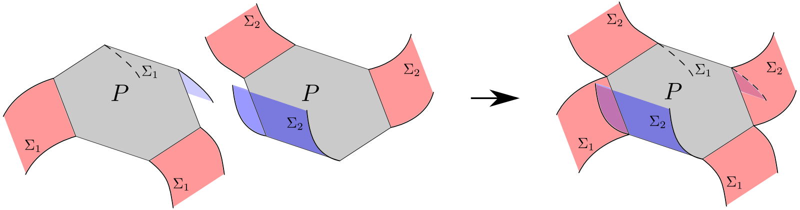
Structure of homogeneous braids

1. Homogeneous braids are fibered $\rightsquigarrow$ Open books $\rightsquigarrow$ Fixed arcs.
2. Homogeneous braids are Murasugi sums of positive and negative braids.

Murasugi sums

Definition

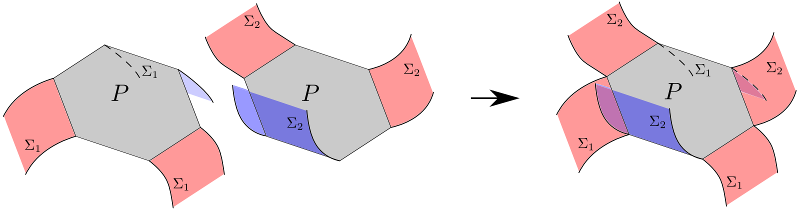
The *Murasugi sum* Σ of two surfaces Σ_1 and Σ_2 along a $2n$–gon P is the surface obtained by gluing Σ_1 to Σ_2 along P. The sum is *essential* if every side of P is non–boundary parallel in Σ.



Murasugi sums

Definition

The *Murasugi sum* Σ of two surfaces Σ_1 and Σ_2 along a $2n$–gon P is the surface obtained by gluing Σ_1 to Σ_2 along P. The sum is *essential* if every side of P is non–boundary parallel in Σ.



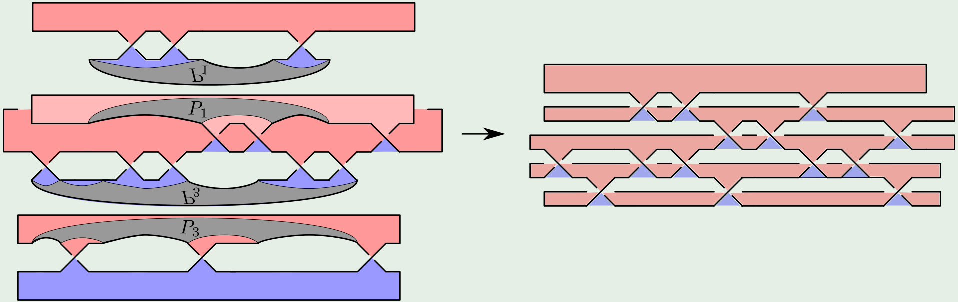
Theorem (Gabai '83)

The surface Σ is a fiber surface if and only if Σ_1 and Σ_2 are fiber surfaces, and moreover its monodromy is $\phi = \phi_2 \circ \phi_1$, where ϕ_i is the monodromy of Σ_i.

Murasugi sums

Homogeneous braids are Murasugi sums of positive and negative braids:

Example



Wishful thinking

Claim. If L_1 and L_2 are prime fibered links, and L their essential Murasugi sum, then L is prime.

Wishful thinking

Claim. If L_1 and L_2 are prime fibered links, and L their essential Murasugi sum, then L is prime.

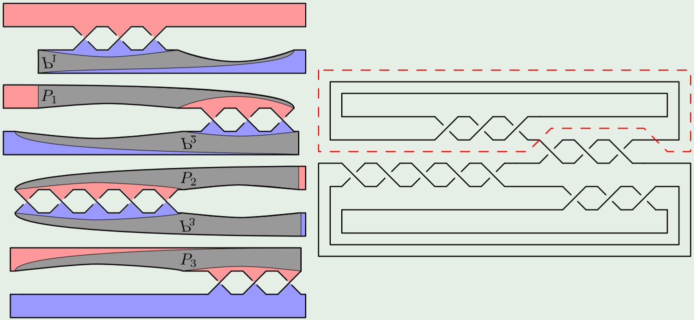
Remark. Being essential is important, as non–essential sums can lead to connected sums.

Wishful thinking

Claim. If L_1 and L_2 are prime fibered links, and L their essential Murasugi sum, then L is prime.

Remark. Being essential is important, as non–essential sums can lead to connected sums.

Example



Wishful thinking

Claim. If L_1 and L_2 are prime fibered links, and L their essential Murasugi sum, then L is prime.

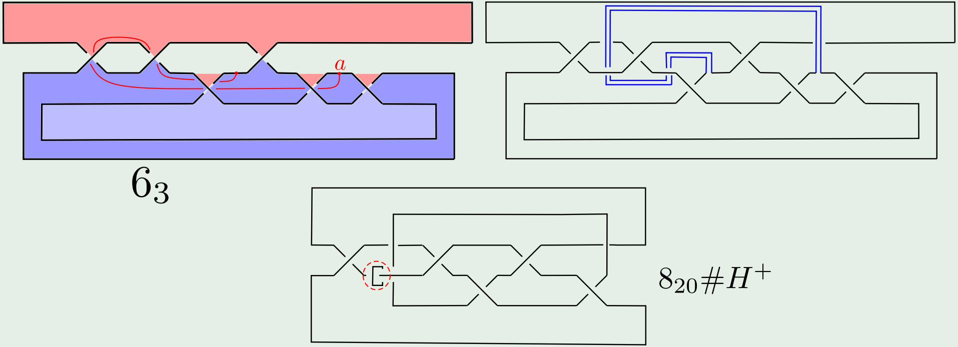
Wishful thinking

Claim. If L_1 and L_2 are prime fibered links, and L their essential Murasugi sum, then L is prime. This is **false**.

Wishful thinking

Claim. If L_1 and L_2 are prime fibered links, and L their essential Murasugi sum, then L is prime. This is **false**.

Example



An open book criterion

Criterion (Feller-Lewark-O. '24)

Let L_1, L_2 be prime fibered links, and L their essential Murasugi sum along P. If Σ_1 is right–veering and $P \subset \Sigma_2$ is left–veering, then L is prime.

Note that by mirroring everything we get an analogous statement when Σ_1 is left–veering and $P \subset \Sigma_2$ is right–veering.

Proof of Criterion: Setup

Lemma

Let Σ *be the essential Murasugi sum of* Σ_1 *and* Σ_2*. If* $\gamma \subset (\Sigma, \phi_2 \circ \phi_1)$ *is a fixed arc, then we can isotope such that* $\phi_1(\gamma) = \phi_2^{-1}(\gamma) := \delta$ *and moreover* $\delta = \gamma$ *outside of* P*.*

Proof of Criterion: Setup

Lemma

Let Σ be the essential Murasugi sum of Σ_1 and Σ_2. If $\gamma \subset (\Sigma, \phi_2 \circ \phi_1)$ is a fixed arc, then we can isotope such that $\phi_1(\gamma) = \phi_2^{-1}(\gamma) := \delta$ and moreover $\delta = \gamma$ outside of P.

Sketch of proof.

First, we have that $\phi_2 \circ \phi_1(\gamma) = \gamma$, so clearly $\phi_1(\gamma) = \phi_2^{-1}(\gamma)$. Call this arc δ.

Proof of Criterion: Setup

Lemma

Let Σ be the essential Murasugi sum of Σ_1 and Σ_2. If $\gamma \subset (\Sigma, \phi_2 \circ \phi_1)$ is a fixed arc, then we can isotope such that $\phi_1(\gamma) = \phi_2^{-1}(\gamma) := \delta$ and moreover $\delta = \gamma$ outside of P.

Sketch of proof.

First, we have that $\phi_2 \circ \phi_1(\gamma) = \gamma$, so clearly $\phi_1(\gamma) = \phi_2^{-1}(\gamma)$. Call this arc δ. Second, in $\Sigma_2 \setminus P$, ϕ_1 is the identity, so $\delta = \phi_1(\gamma) = \gamma$.

Proof of Criterion: Setup

Lemma

Let Σ be the essential Murasugi sum of Σ_1 and Σ_2. If $\gamma \subset (\Sigma, \phi_2 \circ \phi_1)$ is a fixed arc, then we can isotope such that $\phi_1(\gamma) = \phi_2^{-1}(\gamma) := \delta$ and moreover $\delta = \gamma$ outside of P.

Sketch of proof.

First, we have that $\phi_2 \circ \phi_1(\gamma) = \gamma$, so clearly $\phi_1(\gamma) = \phi_2^{-1}(\gamma)$. Call this arc δ. Second, in $\Sigma_2 \setminus P$, ϕ_1 is the identity, so $\delta = \phi_1(\gamma) = \gamma$.Similarly, in $\Sigma_1 \setminus P$, ϕ_2 is the identity, so $\delta = \phi_2^{-1}(\gamma) = \gamma$.

□

Proof of Criterion: Setup

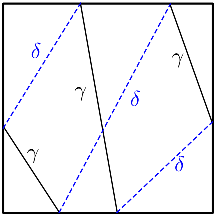
Lemma

Let Σ *be the essential Murasugi sum of* Σ_1 *and* Σ_2*. If* $\gamma \subset (\Sigma, \phi_2 \circ \phi_1)$ *is a fixed arc, then we can isotope such that* $\phi_1(\gamma) = \phi_2^{-1}(\gamma) := \delta$ *and moreover* $\delta = \gamma$ *outside of* P*.*

Proof of Criterion: Setup

Lemma

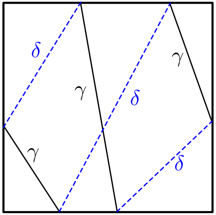
Let Σ be the essential Murasugi sum of Σ_1 and Σ_2. If $\gamma \subset (\Sigma, \phi_2 \circ \phi_1)$ is a fixed arc, then we can isotope such that $\phi_1(\gamma) = \phi_2^{-1}(\gamma) := \delta$ and moreover $\delta = \gamma$ outside of P.



Proof of Criterion: Setup

Lemma

Let Σ be the essential Murasugi sum of Σ_1 and Σ_2. If $\gamma \subset (\Sigma, \phi_2 \circ \phi_1)$ is a fixed arc, then we can isotope such that $\phi_1(\gamma) = \phi_2^{-1}(\gamma) := \delta$ and moreover $\delta = \gamma$ outside of P.



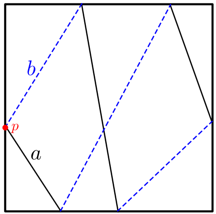
Lemma

In such a disk P, there exists a point p common to chords $a \subset \gamma$ and $b \subset \delta$ such that b is to the left of a at p and the endpoints of a and b are on different sides of P.

Proof of Criterion: Setup

Lemma

Let Σ be the essential Murasugi sum of Σ_1 and Σ_2. If $\gamma \subset (\Sigma, \phi_2 \circ \phi_1)$ is a fixed arc, then we can isotope such that $\phi_1(\gamma) = \phi_2^{-1}(\gamma) := \delta$ and moreover they coincide with γ outside of P.



Lemma

In such a disk P, there exists a point p common to chords $a \subset \gamma$ and $b \subset \delta$ such that b is to the left of a at p, and the endpoints of a and b are on different sides of P.

Proof of Criterion: Setup

Lemma

In such a disk P, there exists a point p common to chords $a \subset \gamma$ and $b \subset \delta$ such that b is to the left of a at p, and the endpoints of a and b are on different sides of P.

Sketch of proof.

Suppose not, that is, at every endpoint the chord from δ is to the right of the chord from γ. Then we get a zigzag

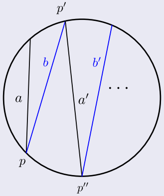
Proof of Criterion: Setup

Lemma

In such a disk P, there exists a point p common to chords $a \subset \gamma$ and $b \subset \delta$ such that b is to the left of a at p, and the endpoints of a and b are on different sides of P.

Sketch of proof.

Suppose not, that is, at every endpoint the chord from δ is to the right of the chord from γ. Then we get a zigzag



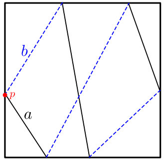
□

Proof of Criterion

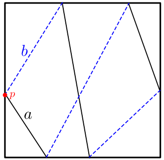
Suppose L is not prime. Then, there exists a fixed arc γ in its fiber surface. Since L_1, L_2 are prime, γ has nontrivial intersection with P.

Proof of Criterion

Suppose L is not prime. Then, there exists a fixed arc γ in its fiber surface. Since L_1, L_2 are prime, γ has nontrivial intersection with P. Then, in P there exist chords $a \subset \gamma$, $b \subset \delta$ such that b is to the left of a at a common endpoint p.



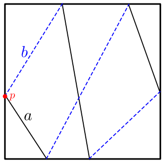
Suppose L is not prime. Then, there exists a fixed arc γ in its fiber surface. Since L_1, L_2 are prime, γ has nontrivial intersection with P. Then, in P there exist chords $a \subset \gamma$, $b \subset \delta$ such that b is to the left of a at a common endpoint p.



Now $p \in \partial\Sigma_i$ for some i, so extend a and b (along γ and δ resp.) to $\tilde{a}$ and $\tilde{b}$ so that they are properly embedded arcs in Σ_i such that $\tilde{b}$ is to the left of $\tilde{a}$.

Proof of Criterion

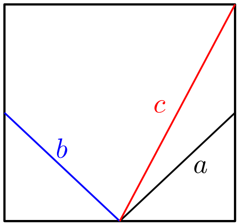
Suppose L is not prime. Then, there exists a fixed arc γ in its fiber surface. Since L_1, L_2 are prime, γ has nontrivial intersection with P. Then, in P there exist chords $a \subset \gamma$, $b \subset \delta$ such that b is to the left of a at a common endpoint p.



Now $p \in \partial\Sigma_i$ for some i, so extend a and b (along γ and δ resp.) to $\tilde{a}$ and $\tilde{b}$ so that they are properly embedded arcs in Σ_i such that $\tilde{b}$ is to the left of $\tilde{a}$.
If $i = 1$, then $\tilde{b} = \phi_1(\tilde{a})$. But by assumption (Σ_1, ϕ_1) is right-veering, contradiction.

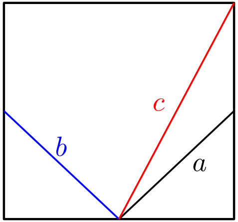
Proof of Criterion

So $i = 2$. Then $\tilde{b} = \phi_2^{-1}(\tilde{a})$. Take an arc $c \subset \Sigma_2$ contained in P that lies between a and b. This is possible because the endpoints of a and b are on different sides of P and the corners of P are on $\partial\Sigma_i$ for $i = 1, 2$.



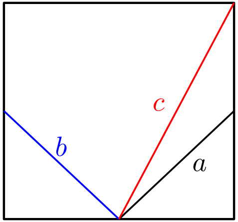
Proof of Criterion

So $i = 2$. Then $\tilde{b} = \phi_2^{-1}(\tilde{a})$. Take an arc $c \subset \Sigma_2$ contained in P that lies between a and b. This is possible because the endpoints of a and b are on different sides of P and the corners of P are on $\partial\Sigma_i$ for $i = 1, 2$.



Since c is to the left of $\tilde{a}$, its image $\phi_2^{-1}(c)$ is to the left of $\tilde{b} = \phi_2^{-1}(\tilde{a})$, which is to the left of c.

So $i = 2$. Then $\tilde{b} = \phi_2^{-1}(\tilde{a})$. Take an arc $c \subset \Sigma_2$ contained in P that lies between a and b. This is possible because the endpoints of a and b are on different sides of P and the corners of P are on $\partial\Sigma_i$ for $i = 1, 2$.



Since c is to the left of $\tilde{a}$, its image $\phi_2^{-1}(c)$ is to the left of $\tilde{b} = \phi_2^{-1}(\tilde{a})$, which is to the left of c. So c is left-veering under ϕ_2^{-1}. However, by assumption ϕ_2 is left-veering in P, which means that ϕ_2^{-1} is right-veering in P, contradiction. □

Proof of Theorem

Criterion

Let L_1, L_2 be prime fibered links, and L their essential Murasugi sum along P. If Σ_1 is right–veering and $P \subset \Sigma_2$ is left–veering, then L is prime.

Theorem

If a homogeneous braid diagram does not have a decomposition circle, then its associated link is prime.

Proof of Theorem

Criterion

Let L_1, L_2 be prime fibered links, and L their essential Murasugi sum along P. If Σ_1 is right–veering and $P \subset \Sigma_2$ is left–veering, then L is prime.

Theorem

If a homogeneous braid diagram does not have a decomposition circle, then its associated link is prime.

Proof.

Let D a homogeneous braid diagram with no decomposition circle.

Proof of Theorem

Criterion

Let L_1, L_2 be prime fibered links, and L their essential Murasugi sum along P. If Σ_1 is right–veering and $P \subset \Sigma_2$ is left–veering, then L is prime.

Theorem

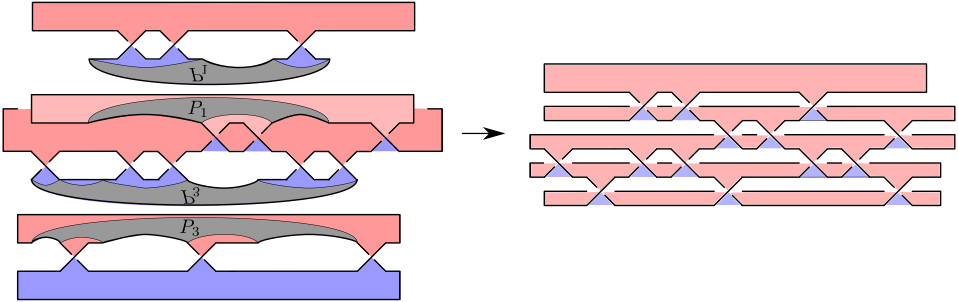
If a homogeneous braid diagram does not have a decomposition circle, then its associated link is prime.

Proof.

Let D a homogeneous braid diagram with no decomposition circle.
L is the Murasugi sum of positive and negative braids, so each sum satisfies the veering properties of the Criterion.

Proof of Theorem

The Murasugi sums satisfy the veering properties of the Criterion.



Proof of Theorem

Criterion

Let L_1, L_2 be prime fibered links, and L their essential Murasugi sum along P. If Σ_1 is right–veering and $P \subset \Sigma_2$ is left–veering, then L is prime.

Theorem

If a homogeneous braid diagram does not have a decomposition circle, then its associated link is prime.

Proof.

Let D a homogeneous braid diagram with no decomposition circle.
L is the Murasugi sum of positive and negative braids, so each sum satisfies the veering properties of the Criterion.

Proof of Theorem

Criterion

Let L_1, L_2 be prime fibered links, and L their essential Murasugi sum along P. If Σ_1 is right–veering and $P \subset \Sigma_2$ is left–veering, then L is prime.

Theorem

If a homogeneous braid diagram does not have a decomposition circle, then its associated link is prime.

Proof.

Let D a homogeneous braid diagram with no decomposition circle.
L is the Murasugi sum of positive and negative braids, so each sum satisfies the veering properties of the Criterion. If there is no decomposition circle, the summands are prime (Cromwell)

Proof of Theorem

Criterion

Let L_1, L_2 be prime fibered links, and L their essential Murasugi sum along P. If Σ_1 is right–veering and $P \subset \Sigma_2$ is left–veering, then L is prime.

Theorem

If a homogeneous braid diagram does not have a decomposition circle, then its associated link is prime.

Proof.

Let D a homogeneous braid diagram with no decomposition circle.
L is the Murasugi sum of positive and negative braids, so each sum satisfies the veering properties of the Criterion. If there is no decomposition circle, the summands are prime (Cromwell) and, moreover, the sums are essential. By the Criterion, L is prime. □

What now?

What now?

There are two directions of work in progress:

What now?

There are two directions of work in progress:

- Improve the Criterion, and find larger classes of links to apply it to.

What now?

There are two directions of work in progress:

- Improve the Criterion, and find larger classes of links to apply it to.
- Use similar methods to extend the results to links that are not fibered. Here we use Gabai's theory of product disks.

Thank you!