FINITENESS OF FOLIATIONS IN BOUNDED GEOMETRY

K-OS, JUNE 19 2025

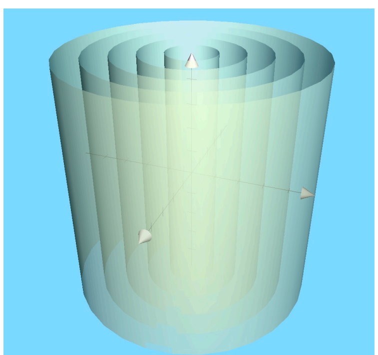
Marco Radeschi

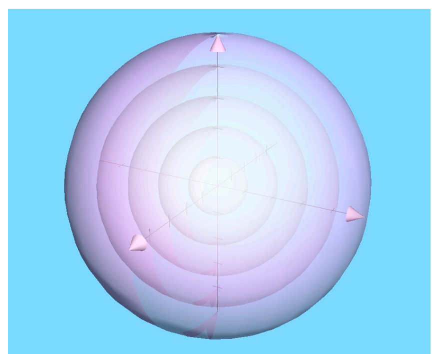
Isoparametric Hypersurfaces

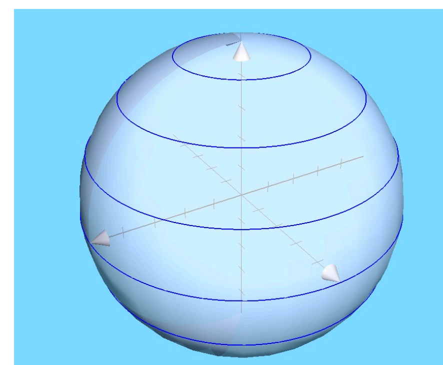
Def: $\bar{M}^n(\kappa)$ space form $(= \mathbb{R}^n, \mathbb{H}^n, \mathbb{S}^n)$, $M \subseteq \bar{M}(\kappa)$ hypersurface

M is *isoparametric*, if:

- principal curvatures are constant.







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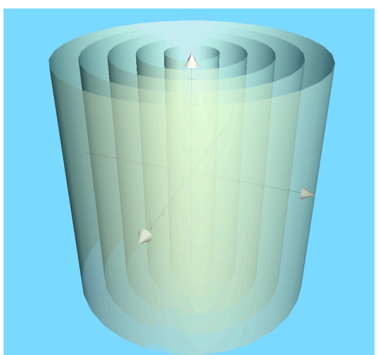

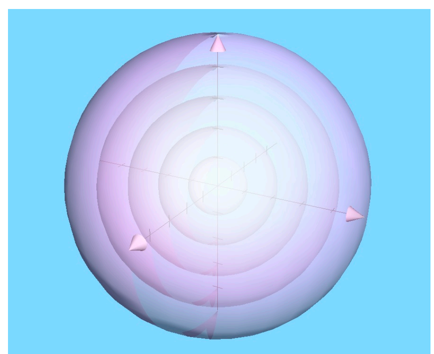

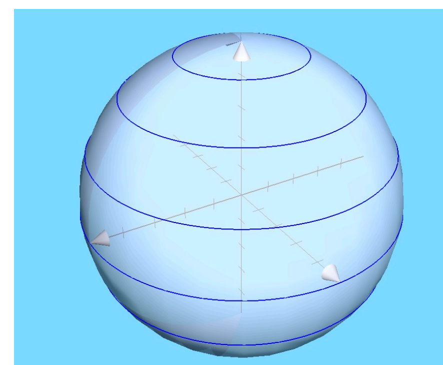

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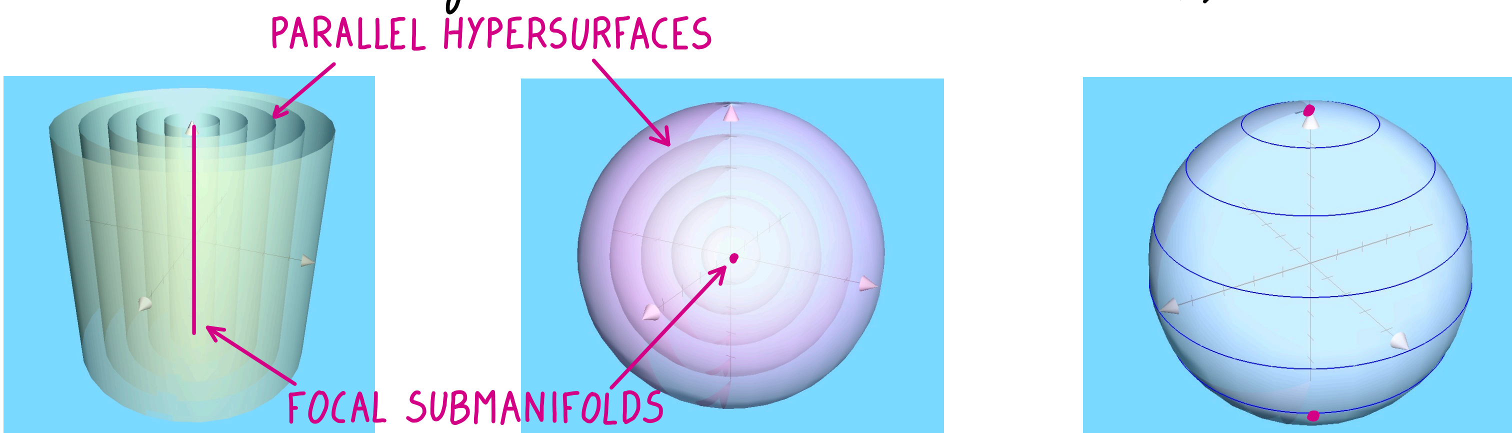

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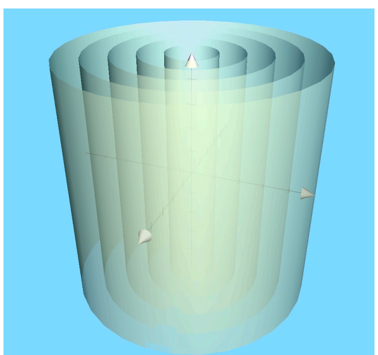

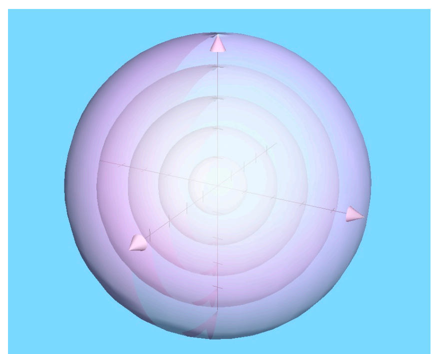

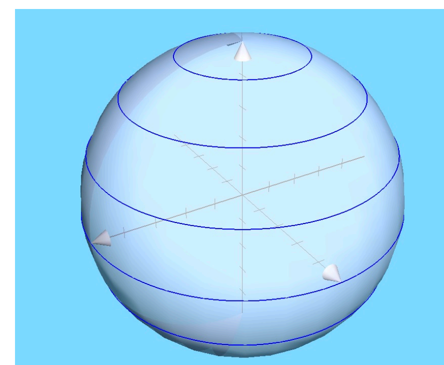

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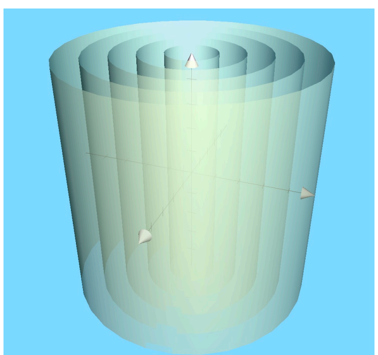

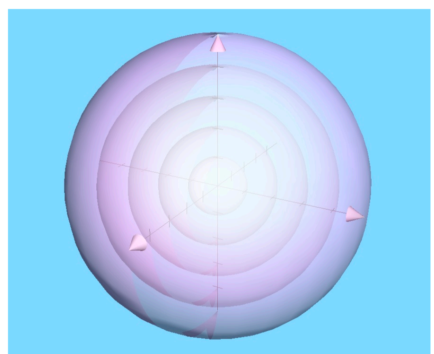

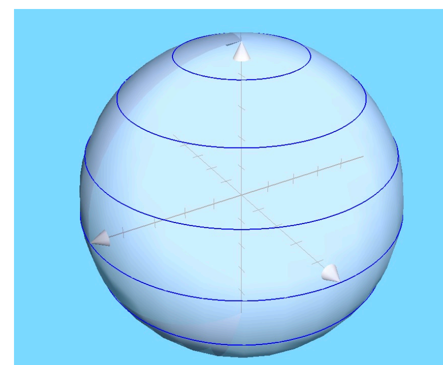

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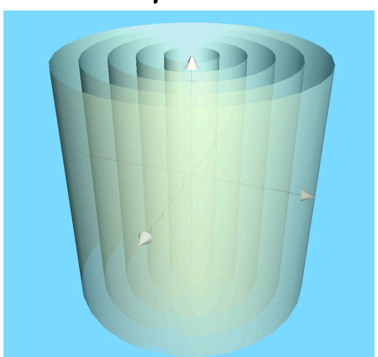
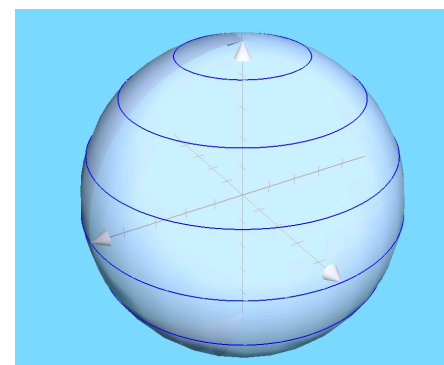
Münzner, Ferus, Karcher, Immervoll, Abresch, Stolz, Miyaoka, Cecil, Dorfmeister-Neher, Chi, Siffert, Jen

almost complete classification of isoparam. hypersurfaces in $\mathbb{S}^n$





The compact case

Def: (M,g) Riemannian manifold. An isoparametric foliation is the partition into level sets of $f: M \longrightarrow [-\ell,\ell]$ s.t. $\|\nabla f\|^2 = b(f)$, $\Delta f = a(f)$

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Rmk: in all cases, there are finitely many foliations

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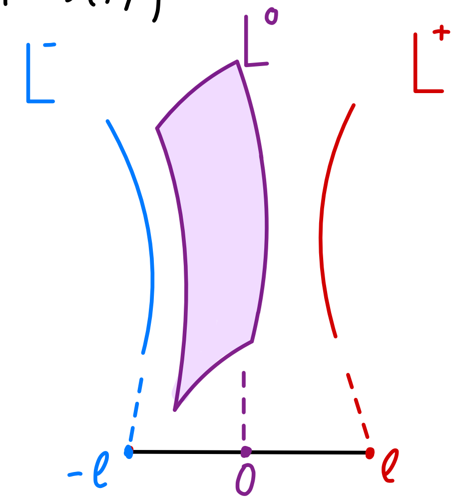

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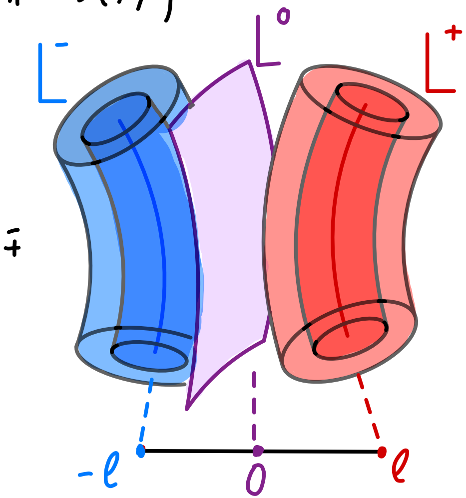

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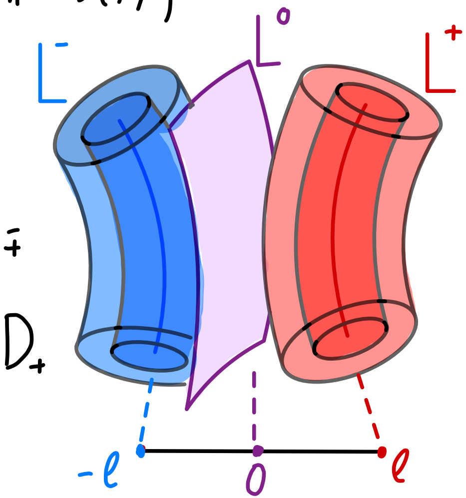

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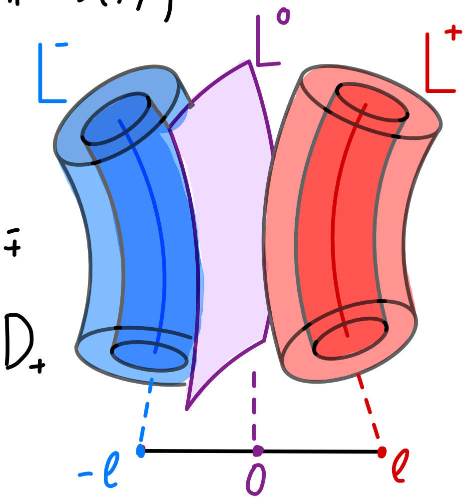

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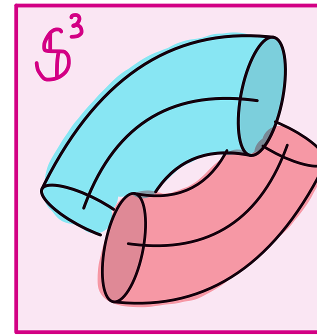

Es: $\mathbb{S}^{n+m-1} \simeq \mathbb{S}^{n-1} * \mathbb{S}^{m-1} \xrightarrow{f} \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

$\cos\theta\, p + \sin\theta\, q \longmapsto \theta - \frac{\pi}{4}$

$\Longrightarrow \begin{cases} L_- \simeq \mathbb{S}^{n-1},\quad L_+ \simeq \mathbb{S}^{m-1},\quad L_0 \simeq \mathbb{S}^{n-1} \times \mathbb{S}^{m-1} \\ \mathbb{S}^{m+n-1} \simeq \mathbb{S}^{n-1} \times \mathbb{D}^m \coprod_{id} \mathbb{D}^n \times \mathbb{S}^{m-1} \end{cases}$

Finiteness Results

Def. $\mathcal{M}_{v}^{K,D}(n) = \{ f : M \longrightarrow [-\ell, \ell] \mid \dim M = n, \pi_1(M) = 0, |\sec M| < K, \operatorname{vol} M < v, \operatorname{diam} M < D \}$

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$f_i : M_i \to [-\ell_i, \ell_i]$, $i = 1,2$ are **foliated diffeomorphic** if $\exists$

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- On $M = S^n = S^1 * S^{n-2}$, $n \geq 5$, $\exists \infty$ $f_i : (M, g_i) \to [-1,1]$ with $L_- = S^1$, $L_+ = S^n$, $L_0 = S^1 \times S^n$
- On $M = S^3 \times S^4 \times S^5$ $\exists \infty$ $f_i : (M, g_i) \to [-1,1]$ s.t. $L_\pm^i := f_i^{-1}(\mp \ell_i)$ mutually not homeo.

An Equivalence

Given Euclidean disk bundles $D^{\mp} \to L^{\mp}$ and $\varphi \in \mathrm{Diff}(\partial D^-, \partial D^+)$, define

$$\boxed{f_\varphi : D^- \coprod_\varphi D^+ \to [-1,1], \quad f_\varphi(x) = \begin{cases} \|x\| - 1 & x \in D^- \\ 1 - \|x\| & x \in D^+ \end{cases}}$$

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Theorem (Ge, '15):

$G := \pi_0(\mathrm{Diff}_{\ell in}(D^-)) \times \pi_0(\mathrm{Diff}_{\ell in}(D^+))$ acts on $\pi_0(\mathrm{Diff}(\partial D^-, \partial D^+))$ by

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$f_i : D^- \underset{\varphi_i}{\amalg} D^+ \longrightarrow [-1,1]$ satisfy $f_{\varphi_0} \sim f_{\varphi_1}$ iff $[\varphi_0] = [\varphi_1] \in \pi_0(\mathrm{Diff}(\partial D^-, \partial D^+)) / G$

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$(n \neq 5)$

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Proof (idea):

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1. Supp $\exists \{(M_i, f_i)\}_i$, $f_i \not\sim f_j$. $\rightsquigarrow M_i \cong D_i^- \coprod_{\varphi_i} D_i^+$, $\pi_i^\pm : D_i^\pm \to L_i^\pm$

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THE FINITENESS

$(n \neq 5)$

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7. $n-1 \neq 4$: $\text{forget}_* : \pi_0(\text{Diff}) \longrightarrow \pi_0(\text{Homeo})$ has finite fibers $\Rightarrow$ $[\varphi_i'] = [\varphi_j'] \in \pi_0(\text{Diff})$ $\Rightarrow$ $M_i \xrightarrow[\text{diff.}]{\cong} M_j$

Counterexamples

Theorem (Krannich, Lytchak, R. '25):

On $M = \mathbb{S}^n = \mathbb{S}^1 * \mathbb{S}^{n-2}$ $(n \geq 5)$ $\exists \infty$ $f_i : (M, g_i) \to [-1, 1]$ with $L_- = \mathbb{S}^1$, $L_+ = \mathbb{S}^n$, $L_0 = \mathbb{S}^1 \times \mathbb{S}^n$

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COUNTEREXAMPLES

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$\boxed{\exists H \in \mathrm{Diff}(\partial D^- \times I \to \partial D^- \times I) \text{ s.t. } \varphi_i = H|_{\partial D^- \times \{i\}}}$

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$f_{\varphi_i} \not\sim f_{\varphi_j}$

$\exists \Psi : M_i \to M_j$ s.t. $\Psi(f_{\varphi_i}^{-1}(\mp 1)) = f_{\varphi_j}^{-1}(\mp 1)$

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Theorem (Krannich, Lytchak, R. '25):

On $M = \mathbb{S}^3 \times \mathbb{S}^4 \times \mathbb{S}^5$ $\exists \infty$ $f_i : M \to [-1,1]$ w/ $L_i := f_i^{-1}(-1)$ pairwise non-homeo.

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1. Belegradek: On $E = \mathbb{S}^3 \times \mathbb{S}^4 \times \mathbb{R}^5$ $\exists \infty$ $N_i^7 \subseteq E$ st $E \overset{\text{diff}}{\cong} \nu(N_i)$, N_i non-homeo

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2. On $D = S^3 \times S^4 \times \mathbb{D}^5$ $\exists \infty$ disk bundle structures $\pi_i : D \to N_i$

COUNTEREXAMPLES

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3. Each $\pi_i : D \to N_i$ induces $f_i : D \to [0,1]$, $f_i(x) = 1 - \|x\|^2_{g_i}$
4. $M = D^{(1)} \coprod_{id} D^{(2)} \simeq \mathbb{S}^3 \times \mathbb{S}^4 \times \mathbb{S}^5$ carries ∞ distinct $f_{ij} : M \to [-1,1]$ $f_{ij}(x) = \begin{cases} f_i(x) & \text{if } x \in D^{(1)} \\ f_j(x) & \text{if } x \in D^{(2)} \end{cases}$

OPEN QUESTIONS

1. $|\mathcal{M}^{K,D}_{v}(5)| < \infty$? Does $\text{forget}_*: \pi_0(\text{Diff}(M^4)) \to \pi_0(\text{Lip}(M^4))$ have finite Ker?

2. Conjecture: $\{f: M^n \longrightarrow X^m \text{ submetry} \mid |\sec M| < K, \text{vol}\, M > v_1, \text{diam}\, M < D, \text{vol}\, X \geq v_2\}$ contains finitely many foliated homeomorphism types

Thank you!