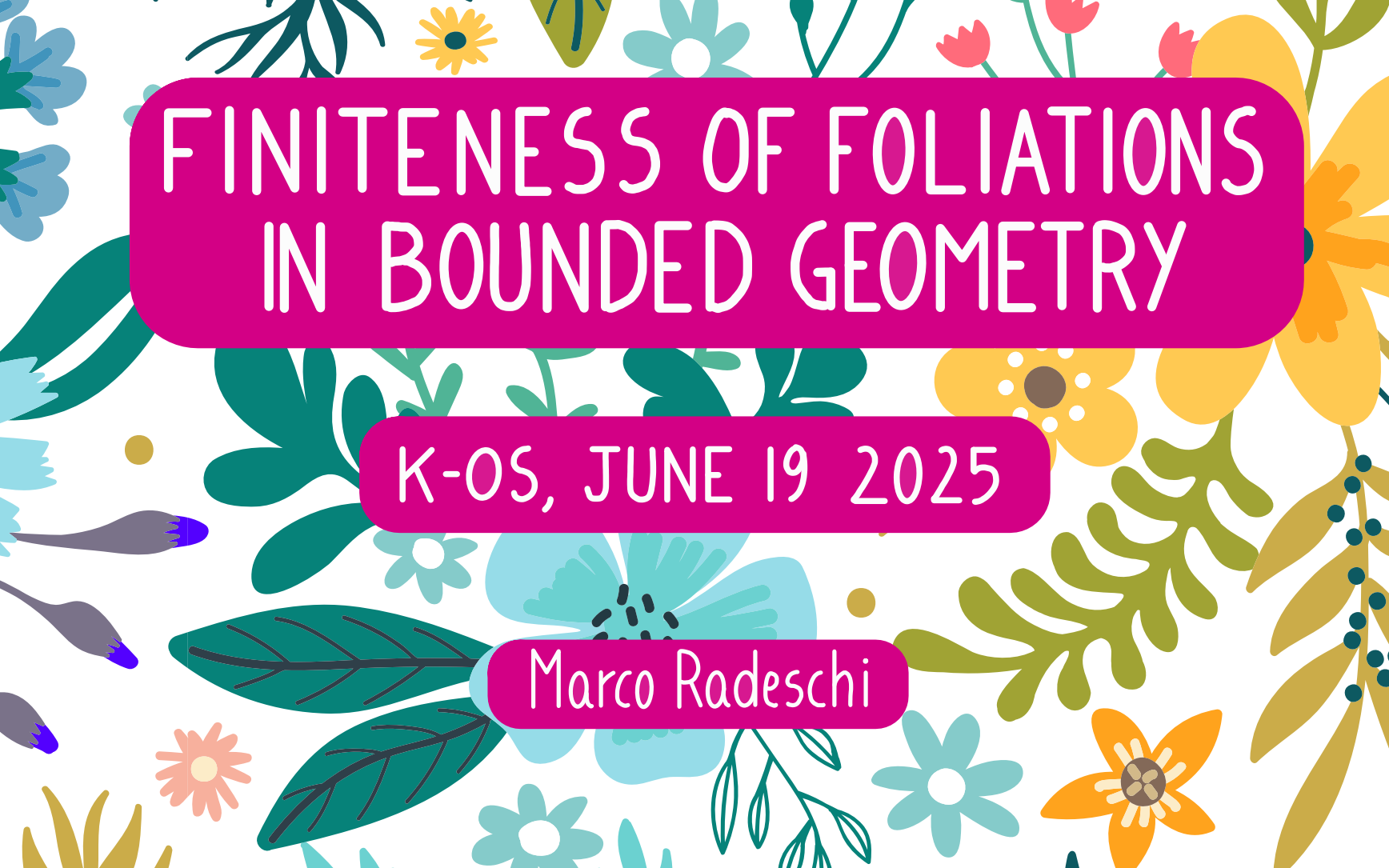


The background of the slide is a vibrant, repeating floral pattern. It features various stylized flowers in shades of blue, yellow, orange, and pink, interspersed with green leaves and stems. The pattern is dense and covers the entire slide area.

FINITENESS OF FOLIATIONS IN BOUNDED GEOMETRY

K-OS, JUNE 19 2025

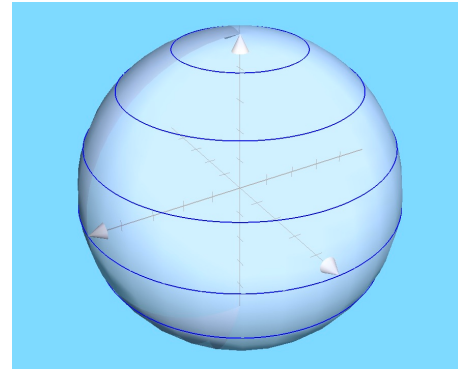
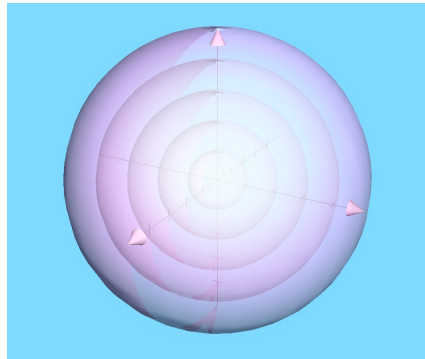
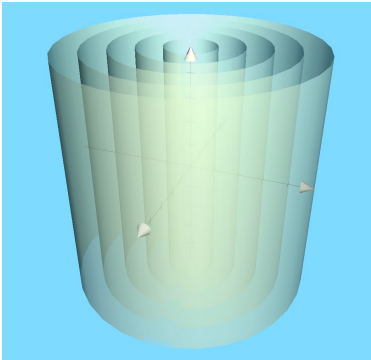
Marco Radeschi

ISOPARAMETRIC HYPERSURFACES

Def: $M(\kappa)$ space form $(= \mathbb{R}^n, \mathbb{H}^n, \mathbb{S}^n)$, $M \subseteq M(\kappa)$ hypersurface

M is *isoparametric*, if:

- principal curvatures are constant.



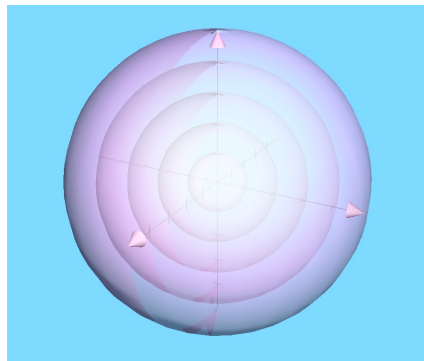
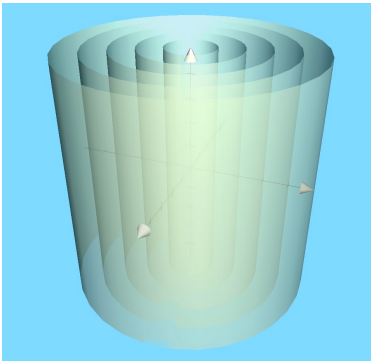
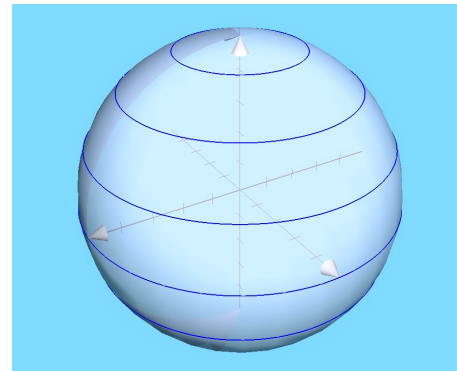
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A 3D visualization of a family of concentric spherical hypersurfaces in Euclidean space. The spheres are nested and centered on a common point. They are rendered with a semi-transparent, light purple/pink color, showing the interior of the outer spheres. Small white arrows point outwards from the surface of the spheres, representing the normal vector field.

ISOPARAMETRIC HYPERSURFACES

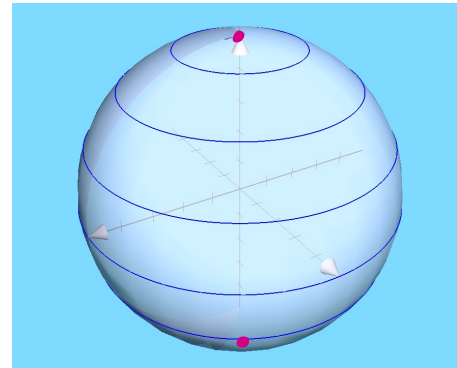
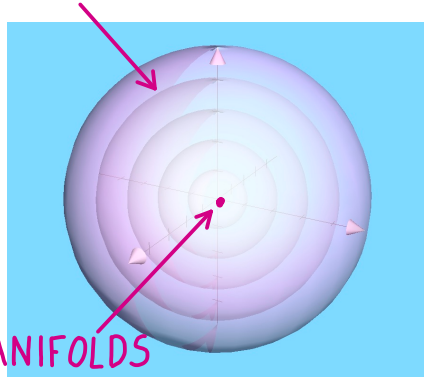
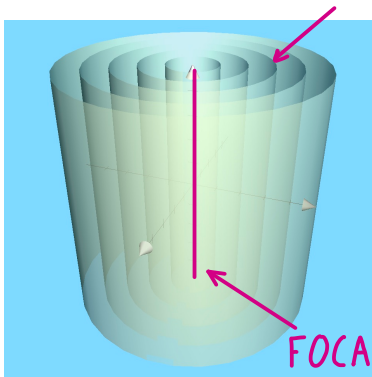
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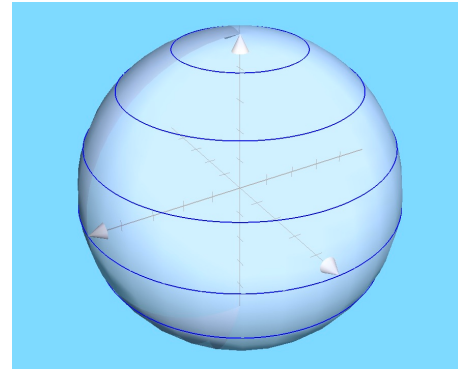
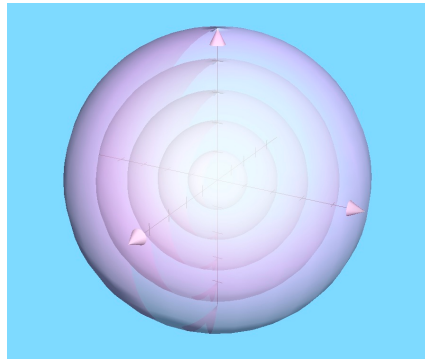
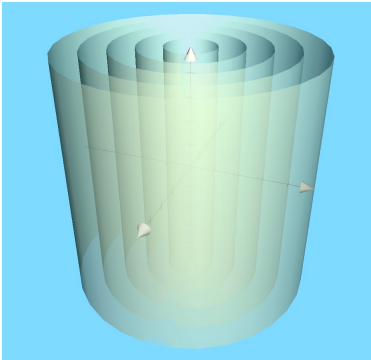
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PARALLEL HYPERSURFACES



ISOPARAMETRIC HYPERSURFACES

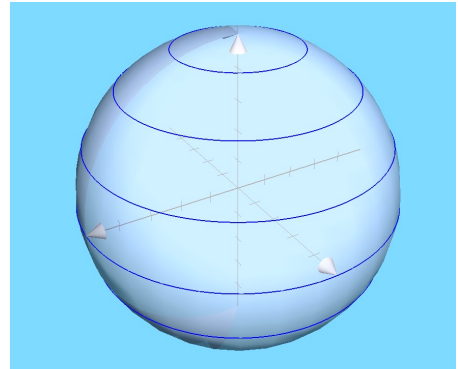
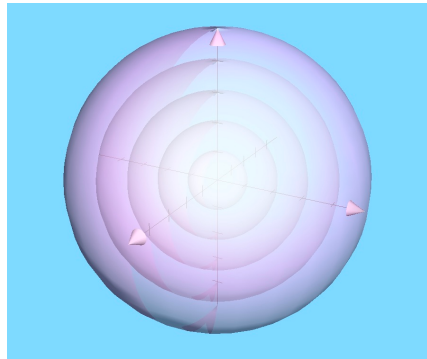
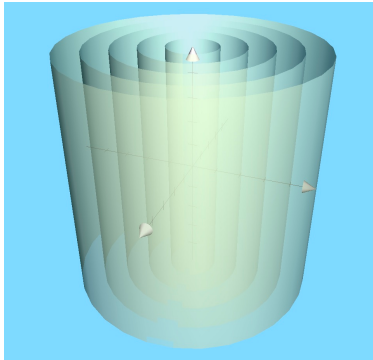
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 $n = 4, 7, 13$, $f = \text{distance from } \mathbb{K}P^2 \subseteq S^n$ ($\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$)

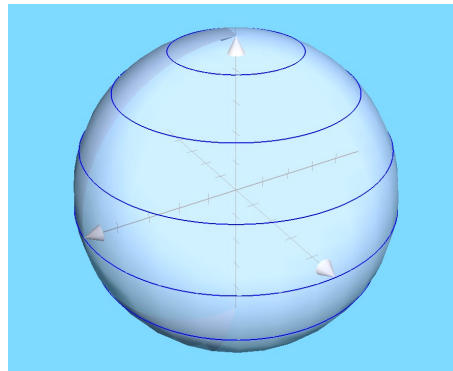
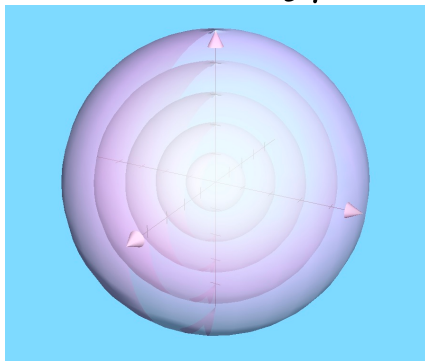
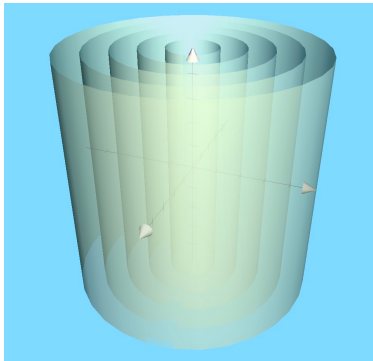


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almost complete classification of isoparam. hypersurfaces in $\mathbb{S}^n$



THE COMPACT CASE

Def: (M, g) Riemannian manifold. An *isoparametric foliation* is the partition into level sets of $f: M \rightarrow [\ell, \ell]$ s.t. $\|\nabla f\|^2 = b(f)$, $\Delta f = a(f)$

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Rmk: in all cases, there are finitely many foliations

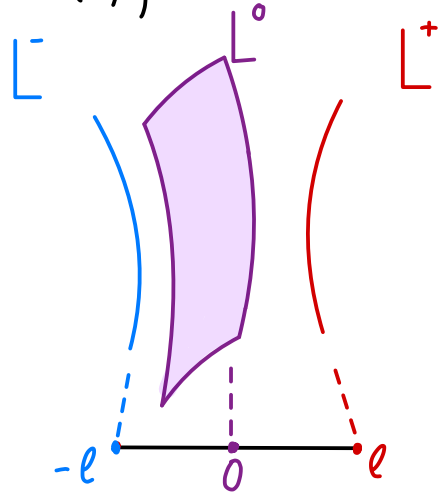
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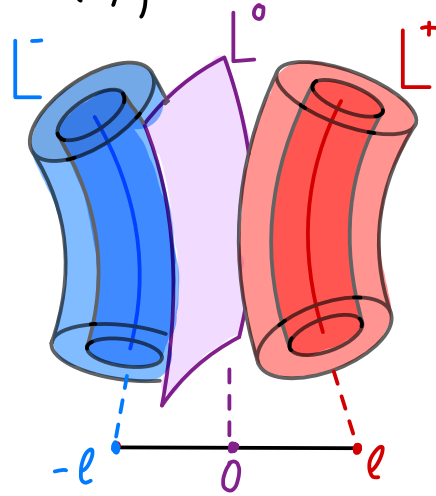


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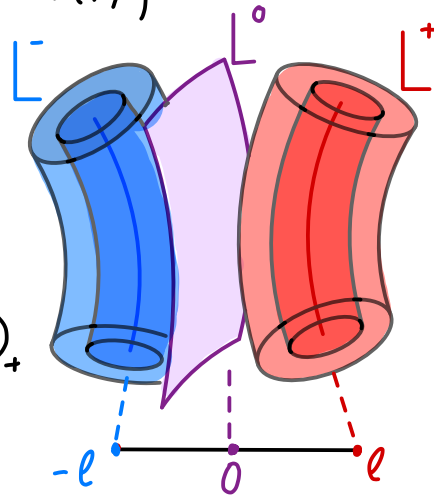
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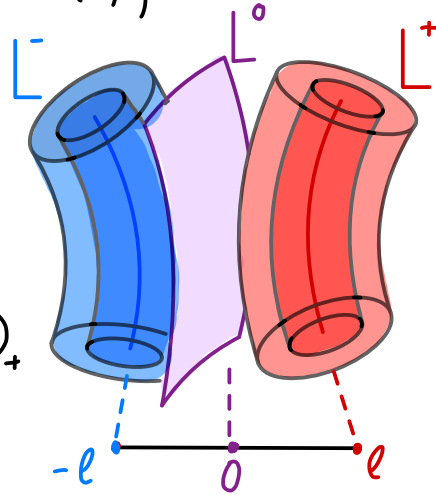
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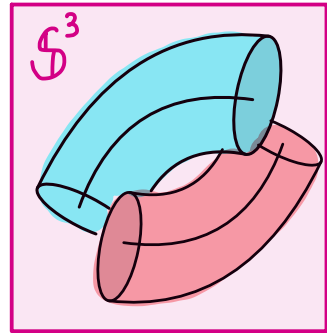
$f: M \rightarrow [-l, l]$ isoparametric function ($\Delta f = a(f)$, $\|\nabla f\|^2 = b(f)$)

Fact: $L_- := f^{-1}(-l)$, $L_+ := f^{-1}(l)$ smooth submanifolds

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$$\text{Es: } S^{n+m-1} \cong S^{n-1} * S^{m-1} \xrightarrow{f} [-\frac{\pi}{4}, \frac{\pi}{4}] \quad \cos \theta p + \sin \theta q \mapsto \theta - \frac{\pi}{4} \quad \Rightarrow \quad \begin{cases} L_- \cong S^{n-1}, & L_+ \cong S^{m-1}, & L_0 \cong S^{n-1} \times S^{m-1} \\ S^{m+n-1} \cong S^{n-1} \times D^m \bigsqcup_{\text{id}} D^n \times S^{m-1} \end{cases}$$

FINITENESS RESULTS

Def. $\mathcal{M}_{\sigma}^{K,D}(n) = \{ f: M \rightarrow [-\ell, \ell] \mid \dim M = n, \pi_1(M) = 0, |\sec M| < K, \text{vol } M < \sigma, \text{diam } M < D \}$

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$$\begin{array}{ccc} M_1 & \xrightarrow{\varphi} & M_2 \\ f_1 \downarrow & & \downarrow f_2 \\ [-l_1, l_1] & \xrightarrow{\varphi_*} & [-l_2, l_2] \end{array}$$

write $f_1 \sim f_2$.

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. On $M = S^3 \times S^4 \times S^5 \exists \infty f_i: (M, g_i) \rightarrow [-1, 1]$ s.t. $L_{\pm}^i := f_i^{-1}(\pm l_i)$ mutually not homeo.

AN EQUIVALENCE

Given Euclidean disk bundles $D^{\pm} \rightarrow L^{\pm}$ and $\varphi \in \text{Diff}(\partial D^-, \partial D^+)$, define

$$f_{\varphi}: D^- \amalg_{\varphi} D^+ \rightarrow [-1, 1], \quad f_{\varphi}(x) = \begin{cases} \|x\| - 1 & x \in D^- \\ 1 - \|x\| & x \in D^+ \end{cases}$$

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$f_i: D^- \amalg_{\varphi_i} D^+ \rightarrow [-1, 1]$ satisfy $f_{\varphi_0} \sim f_{\varphi_1}$ iff $[\varphi_0] = [\varphi_1] \in \pi_0(\text{Diff}(\partial D^-, \partial D^+)) / G$

THE FINITENESS

($n \neq 5$)

Theorem (Krannich, Lytchak, R. '25): $\left| \left\{ f: M^n \rightarrow [-\ell, \ell] \mid \begin{array}{l} \pi, M=0, \\ |sec| < K, \end{array} \begin{array}{l} \text{diam} < D \\ \text{Vol} > \nu \end{array} \right\} / \sim \right| < \infty$

Proof (idea):

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2. Cheeger: $d + sec + vol \Rightarrow \psi_i: M_i \xrightarrow{\text{diff.}} M$ fixed mfd w/ a $C^{1,\alpha}$ -metric

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($n \neq 5$)

Theorem (Krannich, Lytchak, R. '25): $\left| \left\{ f: M^n \rightarrow [-\ell, \ell] \mid \begin{array}{l} \pi_1 M = 0, \text{ diam} < D \\ |sec| < K, \text{ Vol} > \nu \end{array} \right\} / \sim \right| < \infty$

Proof (idea):

1. Supp $\exists \{(M_i, f_i)\}_i, f_i \not\sim f_j. \rightsquigarrow M_i \simeq D_i^- \amalg_{\varphi_i} D_i^+, \pi_i^\pm: D_i^\pm \rightarrow L_i^\pm$
2. Cheeger: $d + sec + vol \Rightarrow \psi_i: M_i \xrightarrow{\text{diff.}} M$ fixed mfd w/ a $C^{1,\alpha}$ -metric
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COUNTEREXAMPLES

Theorem (Krannich, Lylchak, R. '25):

On $M = S^n = S^1 * S^{n-2}$ ($n \geq 5$) $\exists \infty f_i: (M, g_i) \rightarrow [-1, 1]$ with $L_- = S^1, L_+ = S^n, L_0 = S^1 \times S^n$

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$$f_{\varphi_i} \not\sim f_{\varphi_j}$$

$$\exists \Psi: M_i \rightarrow M_j \text{ s.t. } \Psi(f_{\varphi_i}^{-1}(\pm 1)) = f_{\varphi_j}^{-1}(\pm 1)$$

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On $M = \mathbb{S}^3 \times \mathbb{S}^4 \times \mathbb{S}^5$ $\exists \infty f_i: M \rightarrow [-1, 1]$ w/ $L_i := f_i^{-1}(-1)$ pairwise non-homeo.

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4. $M = D^{(1)} \amalg_{\text{id}} D^{(2)} \simeq \mathbb{S}^3 \times \mathbb{S}^4 \times \mathbb{S}^5$ carries ∞ distinct $f_{ij}: M \rightarrow [-1, 1]$ $f_{ij}(x) = \begin{cases} f_i(x) & \text{if } x \in D^{(1)} \\ f_j(x) & \text{if } x \in D^{(2)} \end{cases}$

OPEN QUESTIONS

1. $|\mathcal{M}_{\sigma}^{K,D}(5)| < \infty$? Does $\text{forget}_*: \pi_0(\text{Diff}(M^4)) \rightarrow \pi_0(\text{Lip}(M^4))$ have finite Ker?
2. **Conjecture:** $\{f: M^n \rightarrow X^m \text{ submetry} \mid |\text{sec } M| < K, \text{vol } M > v_1, \text{diam } M < D, \text{vol } X \geq v_2\}$
contains finitely many foliated homeomorphism types

thank
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