Lefschetz fibrations, Rational blowdowns, and exotic 4-manifolds

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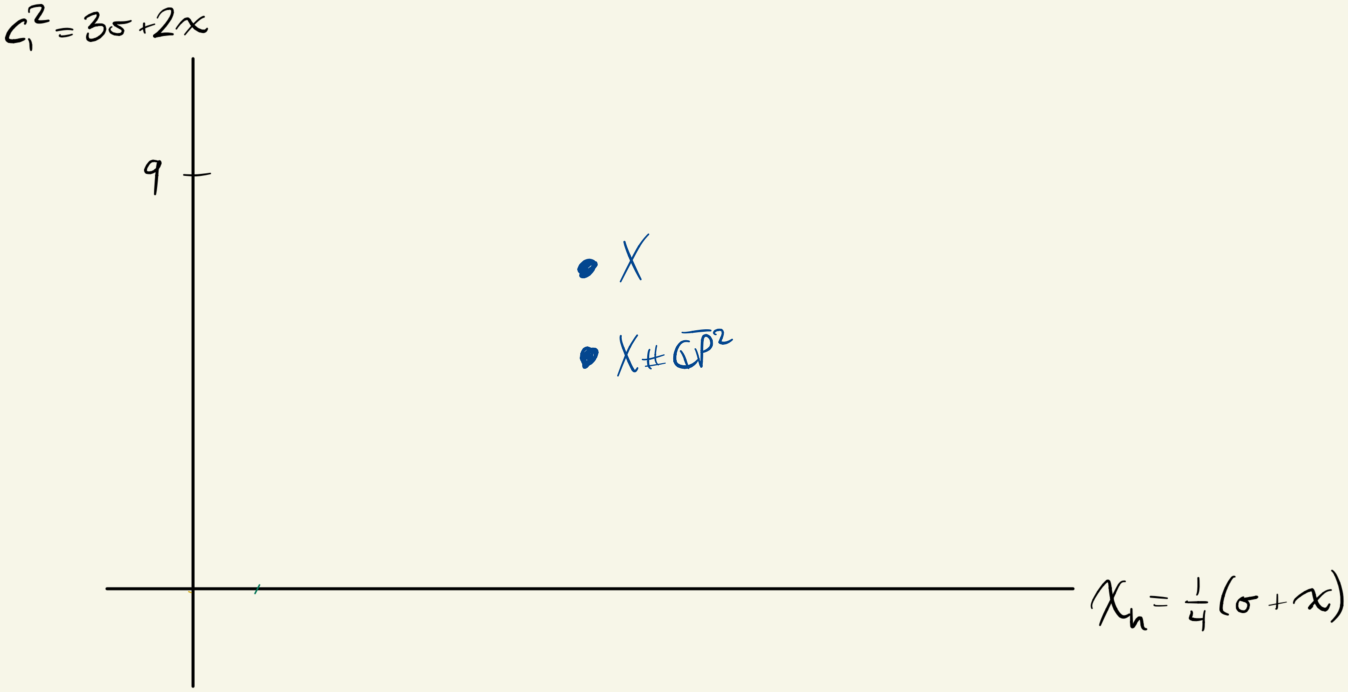
History: Complex Geography Plane

Let X be a complex simply connected closed 4-manifold.



Question: For which values $(a,b) \in \mathbb{Z}^2$ does there exist a simply connected, complex surface X with $\chi_h(X) = a$ and $c_1^2(X) = b$

If X is complex, so is $X \# \overline{\mathbb{CP}^2}$ (blowup of X) and $c_1^2(X \# \overline{\mathbb{CP}^2}) = c_1^2(X) - 1$, $\chi_h(X) = \chi_h(X \# \overline{\mathbb{CP}^2})$

X is called minimal if $X \neq X' \# \overline{\mathbb{CP}^2}$ for some cx X'.

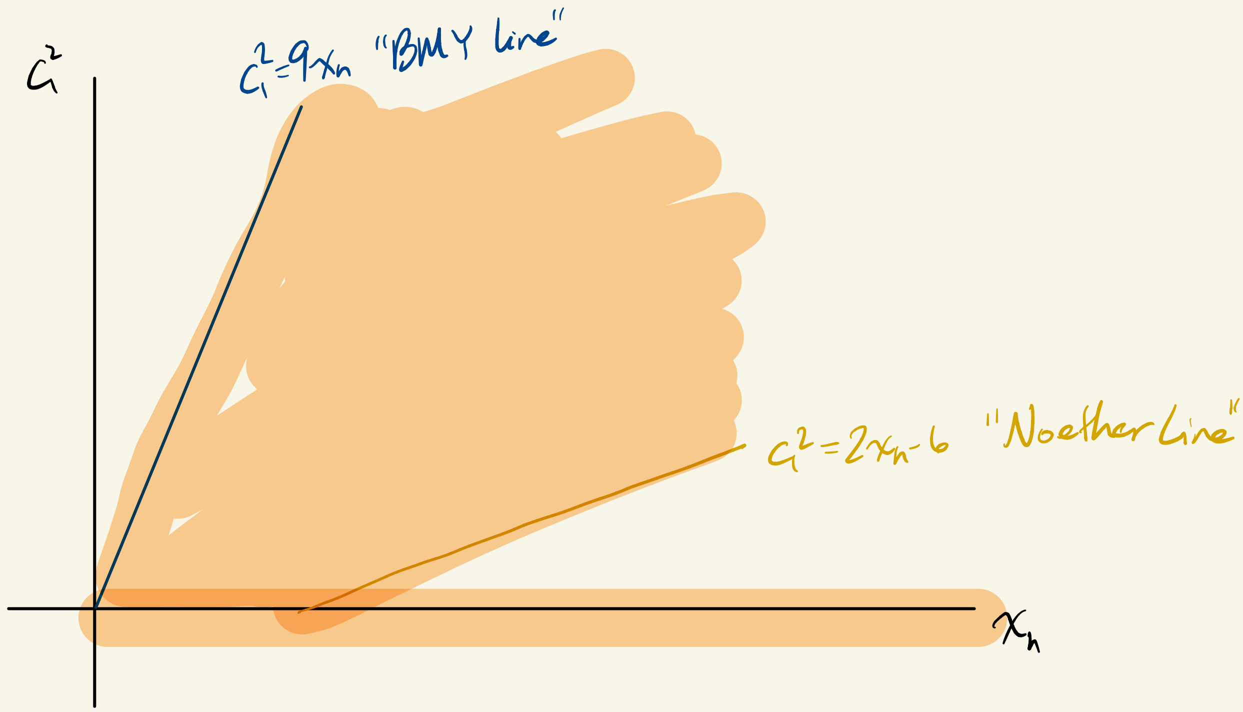
Question': For which values $(a,b) \in \mathbb{Z}^2$ does there exist a simply connected, minimal complex surface X with $\chi_h(X) = a$ and $c_1^2(X) = b$

Theorem: If X is a minimal simply connected complex surface then

$$c_1^2(X) = 0 \text{ and } \chi_h(X) > 0$$

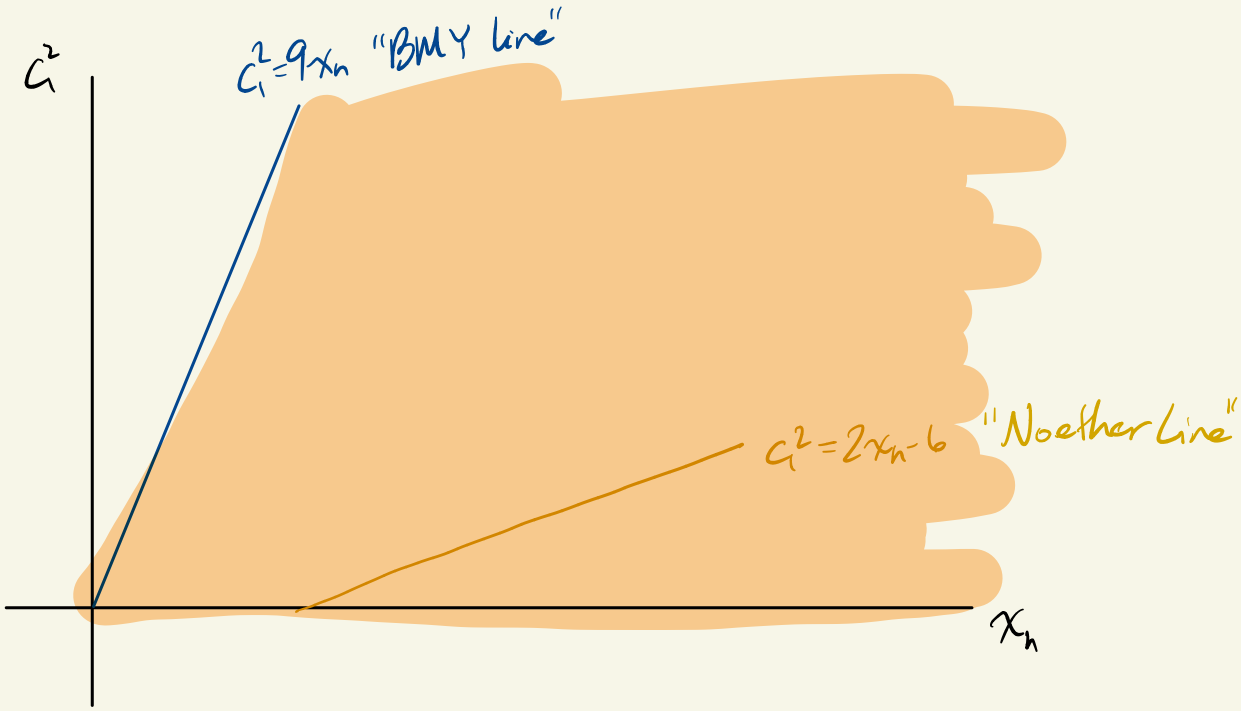
OR

$$c_1^2(X) > 0 \text{ and } 2\chi_h(X) - 6 \le c_1^2(X) \le 9\chi_h(X)$$



Fact: Most Lattice points populated by (holomorphic) Lefschetz fibrations

Symplectic Geography Plane



All simply connected complex surfaces are symplectic. So all lattice points filled in by minimal complex surfaces are filled in minimal symplectic simply connected 4-manifolds.

However, the Noether inequality does not hold (Fintushel-Stern, Gompf, Park, Stipsicz)

These results do not use Lefschetz fibrations (in general)

Theorem 1 (Baykur-Korkmaz-S.)

For each point $(a,b) \in \mathbb{Z}^2_{>0}$ below the Noether line, $\exists$ a minimal simply connected symplectic genus $g = a-1$ Lefschetz fibration with $\chi_h = a$ and $c_1^2 = b$.

Construction: Start with genus $g > 1$ Lefschetz fibrations with clustered nodes,
perform fiber sums, and
perform rational blowdowns

Lefschetz fibrations

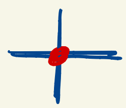
X = compact, oriented, connected, smooth, closed

A Lefschetz fibration on X is a surjective map $f: X \to S^2$ with a finite collection of critical points $B = \{p_1, \dots, p_k\}$ such that:

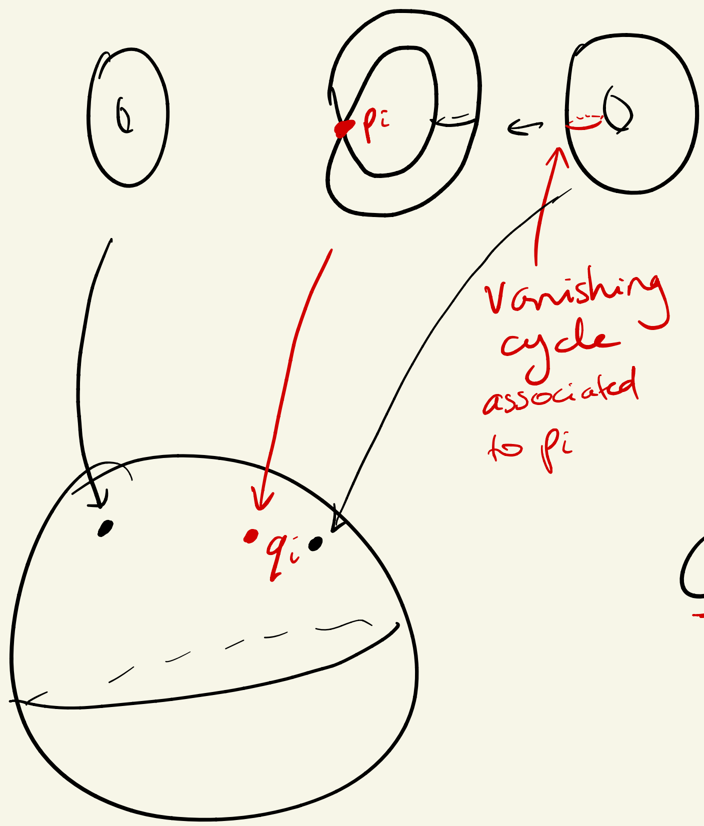
- $f|_{X-B}$ is a locally trivial surface bundle.

  If z is a regular value, $f^{-1}(z) = \Sigma_g$ is a genus g surface

- $\forall\, i$, $\exists$ orientation-preserving complex charts about p_i and $q_i = f(p_i)$ on which $f(z_1, z_2) = z_1 z_2$

  $f^{-1}(0) =$  surface intersecting itself transversely

If a regular fiber has genus g, then (X, f) is called a genus g Lefschetz fibration.

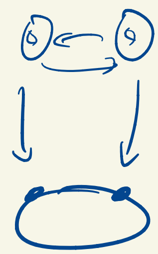


Fact: We can perturb LF to ensure singular fiber contains exactly one singular point

Gompf: Lefschetz fibrations admit symplectic structures with symplectic fibers.

Monodromy Factorization

We can describe a Lefschetz fibration (X, f) combinatorially using monodromy.

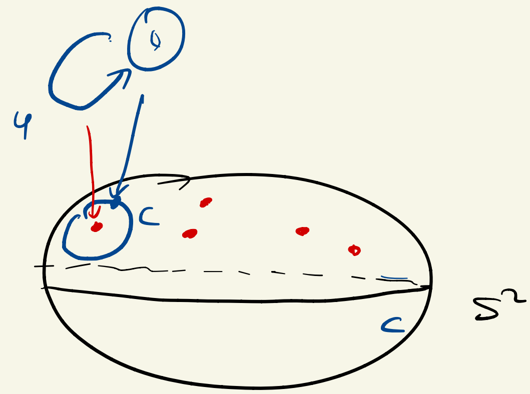


Let C be a circle in S^2 s.t. $C \cap f(B) = \emptyset$

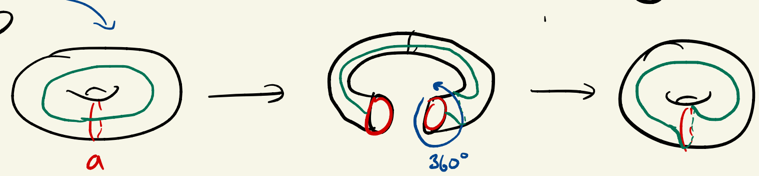
Then $f|_{f^{-1}(C)}$ is a surface bundle over S^1:

$$f^{-1}(C) = \Sigma \times [0,1] \Big/ (x,0) = (\varphi(x), 1)$$

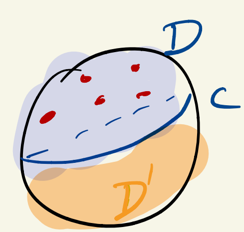
where $\varphi: \Sigma \to \Sigma$ is diffeom called monodromy. $\left(\varphi \in \mathrm{Mod}(\Sigma)\right)$



- If C bds disk w/ $D \cap f(B) = \emptyset$, then $f^{-1}(C) = \Sigma \times S^1$
  $\Rightarrow$ monodromy is 1.
- If C bds disk D w/ $D \cap f(B) = \{p\}$, then $f^{-1}(C)$ has monodromy given by a (positive) Dehn twist about the vanishing cycle associated to p



- If C bds disk D w/ $D \cap f(B) = f(B)$
  Then $f^{-1}(C)$ has monodromy that is a product of monodromies about each crit. pt.
  $\Rightarrow$ product of Dehn twists about vanishing cycles $t_{c_1} t_{c_2} \cdots t_{c_k}$



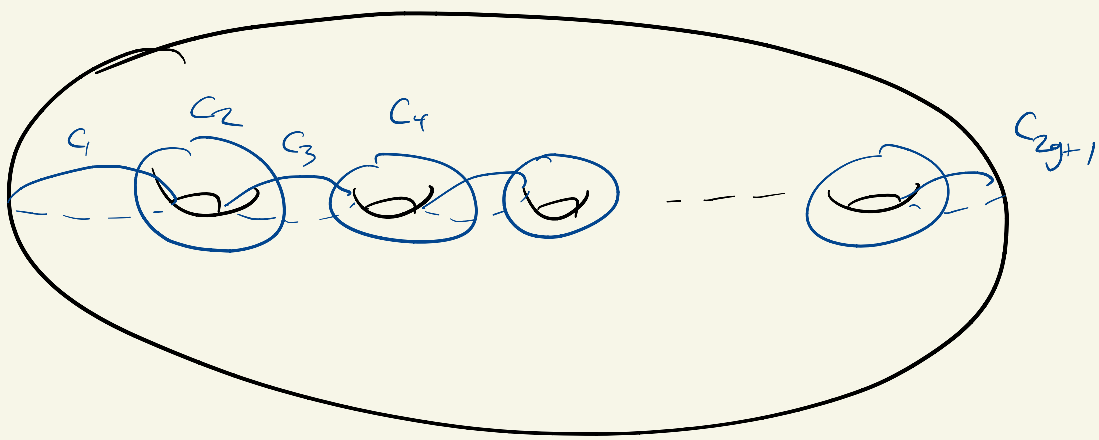
But, C also bds D' s.t. $D' \cap f(B) = \emptyset \Rightarrow t_{c_1} \cdots t_{c_k} = 1 \in \mathrm{Mod}(\Sigma)$

Fact:

- Any Lefschetz fibration can be described by a factorization of the identity in $Mod(\Sigma_g)$ by Dehn twists.
- Conversely, any factorization of the identity in $Mod(\Sigma_g)$ into Dehn twists describes a Lefschetz fibration.

Ex: Hyperelliptic Involution

$\Sigma_g =$



$$\mu = (\underbrace{t_{c_1} \cdots t_{c_{2g+1}} \cdot t_{c_{2g+1}} \cdots t_{1}}_{\text{Hyperelliptic Involution}})^2 = 1$$

Total space of the associated Lefschetz fibration is $\mathbb{CP}^2 \# (4g+5)\overline{\mathbb{CP}^2}$

Constructions of Closed 4-manifolds (w/ LFs)

1. Fiber Sum

Let $(X_1, f_1), (X_2, f_2)$ be genus g Lefschetz fibrations with monodromies $\mu_1 = 1$ and $\mu_2 = 1$

The fiber sum $X_1 \#_f X_2$ is the genus g Lefschetz fibration with monodromy $\mu_1 \mu_2 = 1$.

2. Rational Blowdown (Fintushel-Stern, Park)

Fact: The lens space $L(p^2, pq-1)$, $p > q > 0$, $(p,q) = 1$ bounds the 4-manifolds:

- $C_{p,q} = \overset{-a_1}{\bullet}\!\!-\!\!\!-\!\!\overset{-a_2}{\bullet} - - - \overset{-a_n}{\bullet}$ where $a_1 - \cfrac{1}{a_2 - \cfrac{1}{a_3 - \cdots \frac{1}{a_n}}} = \dfrac{p^2}{pq-1}$

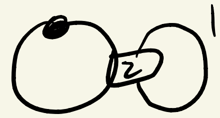
  Plumbing of D^2-bundles over S^2.
  Thickened 2-spheres w/ self int $-a_1, -a_2, \ldots, -a_n$ intersecting according to edges.

- $B_{p,q}$ = a rational homology 4-ball

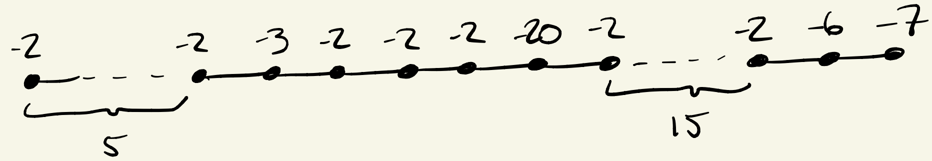
  i.e. $H_*(B_{p,q}; \mathbb{Q}) = H_*(B^4; \mathbb{Q})$

Ex: a) $p=2,\ q=1 \implies \frac{p^2}{pq-1}=4$

$C_{4,1} =$ 

$B_{4,1} =$ 

b) $p=585,\ q=291914$

$C_{p,q} =$ 

Suppose $C_{p,q}$ is embedded in a closed 4-manifold X.
Then $X_{p,q} = (X - C_{p,q}) \cup B_{p,q}$ (doesn't depend on gluing) is obtained by rationally blowing down $C_{p,q}$

Note: $\chi_h(X_{p,q}) = \chi_h(X)$

$c_1^2(X_{p,q}) = c_1^2(X) + n$

Symington: If X is symplectic and $C_{p,q}$ is symplectically embedded, then $X_{p,q}$ is symplectic.

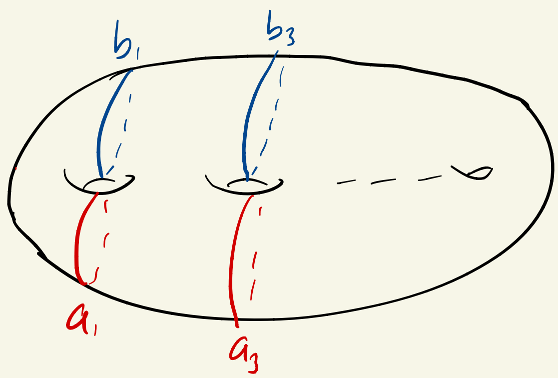
Theorem 1 (Baykur–Korkmaz–S.)

For each point $(a,b) \in \mathbb{Z}^2_{>0}$ below the Noether line, $\exists$ a minimal simply connected symplectic genus $g = a-1$ Lefschetz fibration with $\chi_h = a$ and $c_1^2 = b$.

proof:

Lemma: We can factor the hyperelliptic involution in two ways:

$$\mu_1 = A \cdot t_{a_1}^{2g+2} \cdot t_{a_3}^{2g+2} = 1$$
$$\mu_2 = t_{b_1}^{2g+2} \cdot t_{b_3}^{2g+2} \cdot B = 1$$



Let (X_g, f_g) be the LF w/ monodromy

$$\mu_1\mu_2 = A\, t_{a_1}^{2g+2}\, t_{a_3}^{2g+2}\, t_{b_1}^{2g+2}\, t_{b_3}^{2g+2}\, B = A\,(t_{a_1}t_{a_3}t_{b_1}t_{b_3})^{2g+2}\, B$$

Then X_g is

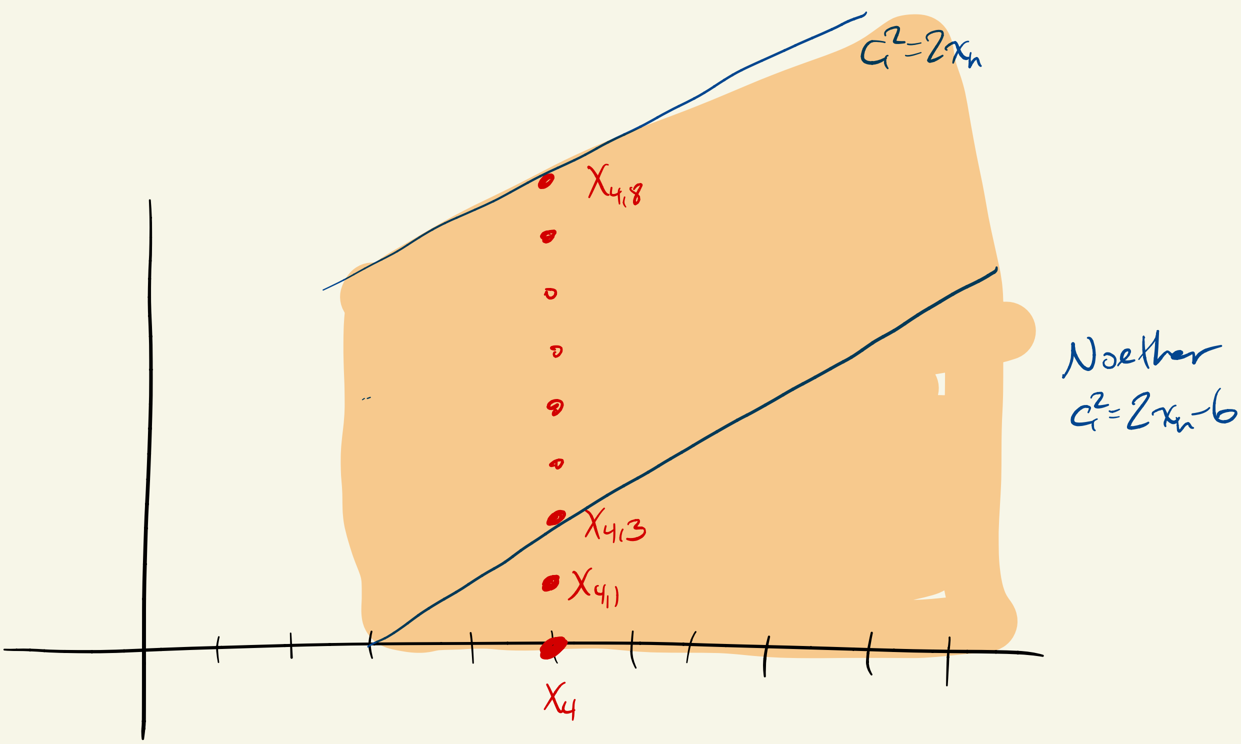
- symplectic (Gompf)
- simply connected (SvK)
- minimal (Usher)

And X_g contains $2g+2$ disjoint -4-spheres

Let $X_{g,r}$ be the result of rationally blowing down r of the -4-spheres.

Then:

- $X_{g,r}$ admits LF (via monodromy substitutions)
- $X_{g,r}$ is simply connected
- $X_{g,r}$ minimal (Dorfmeister)
- $\chi_h(X_{g,r}) = g+1$, $c_1^2(X_{g,r}) = r$ $(0 \le r \le 2g+2)$



$\rightsquigarrow$ Filled up plane below Noether w/ LFs. $\blacksquare$

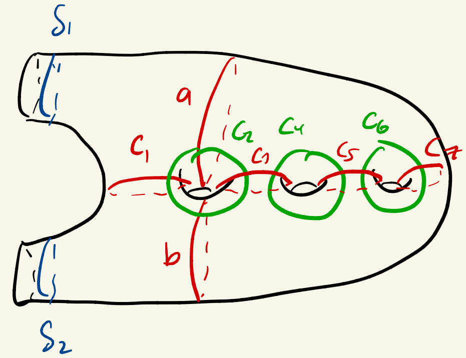
Theorem 2 (Baykur-Korkmaz-S.)

$\exists$ a minimal symplectic exotic $\mathbb{CP}^2 \# 5\overline{\mathbb{CP}^2}$ obtained by rationally blowing down a blowup of a genus 3 Lefschetz fibration

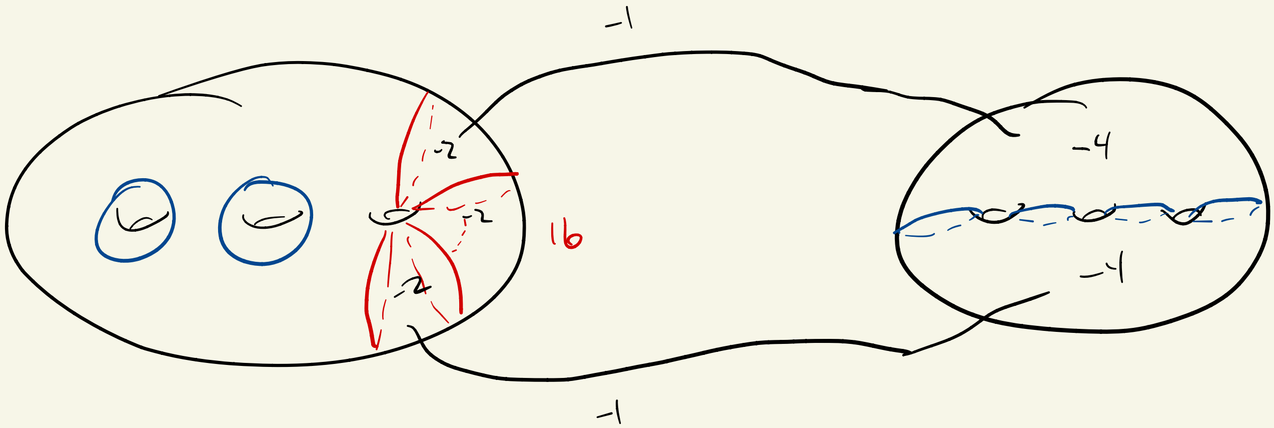
Proof of Theorem 2

Lemma: In $\mathrm{Mod}(\Sigma_3^2)$,

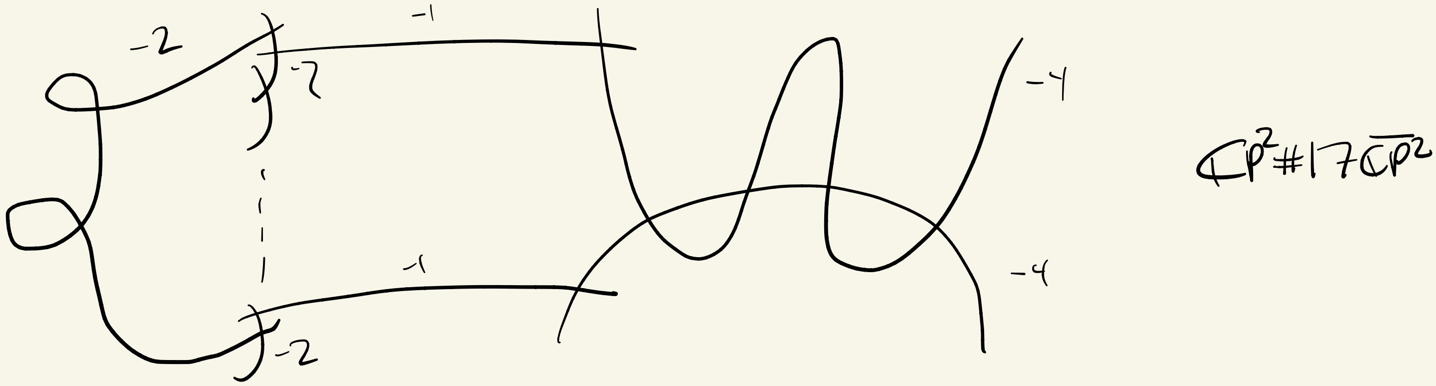
$$t_{\delta_1} \cdot t_{\delta_2} = (t_{c_1}^{14} t_a t_b t_{c_4} t_{c_6}) \cdot (t_{c_4} t_{c_3} t_{c_5} t_{c_7})^{\omega} \cdot D_6$$



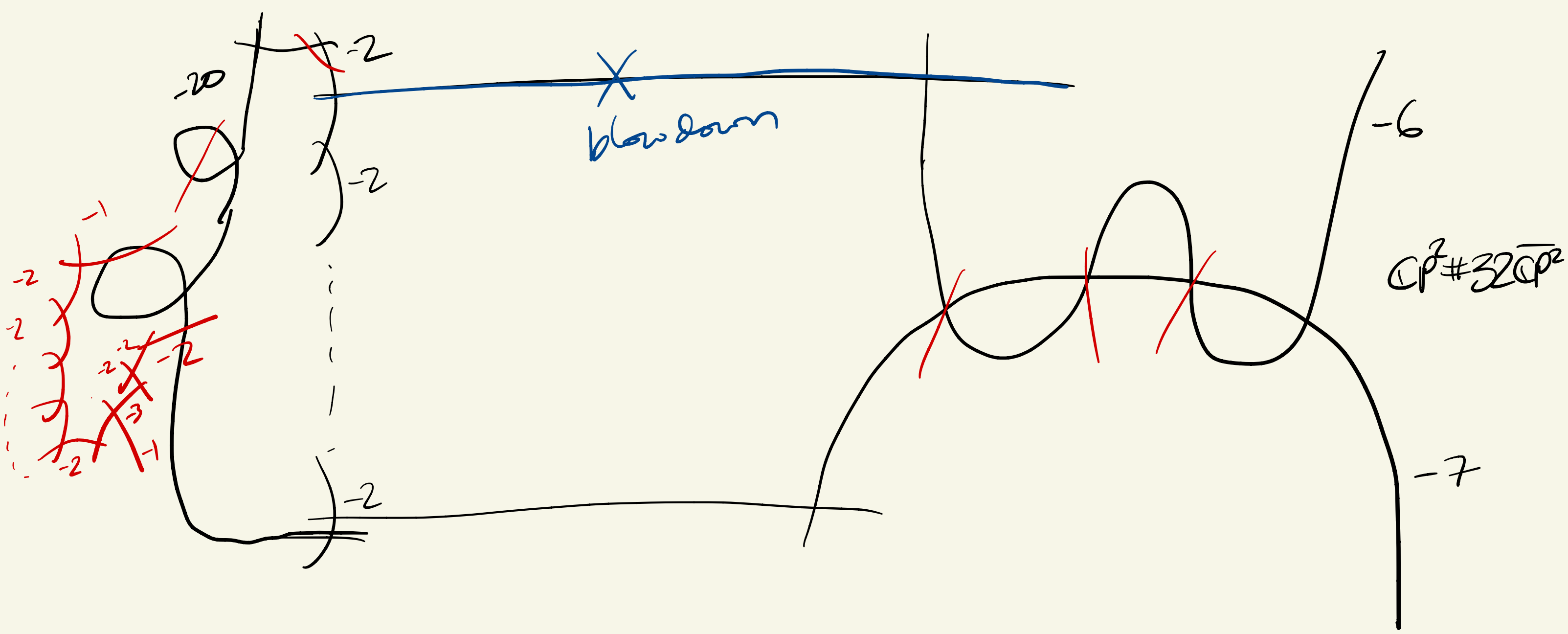
This gives rise to a LF with singular fibers and sections:

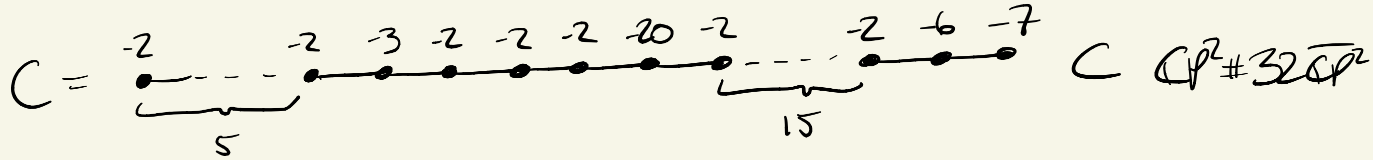


Schematically:



$\downarrow$ blow ups + blow down





Rationally blowing down yields an exotic $\mathbb{C}P^2 \# 5 \overline{\mathbb{C}P^2}$