SU(2)-rep's of toroidal homology spheres

Geneva, 16.6.21

Problem 1: (Kirby's list) — from

Suppose Y is a closed 3-mfld, and $Y \neq S^3$, is there a non-trivial rep'n
$$\pi_1(Y) \longrightarrow SU(2) \ ?$$

Why SU(2)?

Natural from the pt of view of instanton gauge theory.

Problem 2:

Is $I(Y) \neq 0$ if $Y \neq S^3$?

Rk: Problem 1 really is interesting only for $\mathbb{Z}HS^3$'s, i.e. $H_1(Y;\mathbb{Z})=0$.

Known partial results:

- Casson: Yes if $\lambda(Y)\neq 0$.
- Fintushel-Stern: Yes if Y is a Seifert fibred $\mathbb{Z}HS^3$.
- Graph manifold: Yes (Cornwell-Ng-Sivek, Z '15)
- Baldwin-Sivek: Yes for boundaries of Stein domains which are not homology balls

- Kronheimer–Mrowka '03:
  Yes for $S^3_{1/n}(K)$ $n \in \mathbb{Z}_{\neq 0}$
  if $K \subseteq S^3$ is a non-trivial knot

- Z '16: Yes for splicings for non-trivial knots in S^3

  Def: A splicing of knots $K, K' \subseteq S^3$ is

$$Y_{K,K'} = S^3 \setminus N(K) \underset{\substack{\mu \mapsto \ell' \\ \ell \mapsto \mu'}}{\cup} S^3 \setminus N(K')$$

Def: A 3-mfld Y is called toroidal if $\exists\, T^2 \hookrightarrow Y$ which is π_1-injective.

Thm (Lidman, Pinzón Caicedo, Z. '21)

Let Y be a toroidal $\mathbb{Z}HS^3$. Then $\exists$ irreducible

$$\rho : \pi_1(Y) \longrightarrow SU(2)$$

III. Background: pillowcase.

M a mfd

$$R(M) := \{\rho : \pi_1(M) \longrightarrow SU(2)\} / \sim_{\text{conjug}}$$

$$R^w(M) := \{\rho : \pi_1(M) \to SO(3) \mid w_2(\rho) = w\} / \sim_{\text{conjug. in } SO(3)}$$

$$G \xrightarrow{\ \rho\ } SO(3), \qquad SU(2) \xrightarrow{2:1} SO(3), \qquad G \dashrightarrow SU(2)$$

ρ lifts iff $w_2(\rho) = 0$

$R(T^2)$. $\pi_1(T^2) = \mathbb{Z}^2$
$= \langle m, \ell \rangle$

$[\rho] \in R(T^2)$ $\rho: \pi_1(T^2) \to SU(2)$

we may suppose (up to conjug.)

$$\rho(m) = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix} \qquad \alpha \in \mathbb{R}/2\pi\mathbb{Z}$$

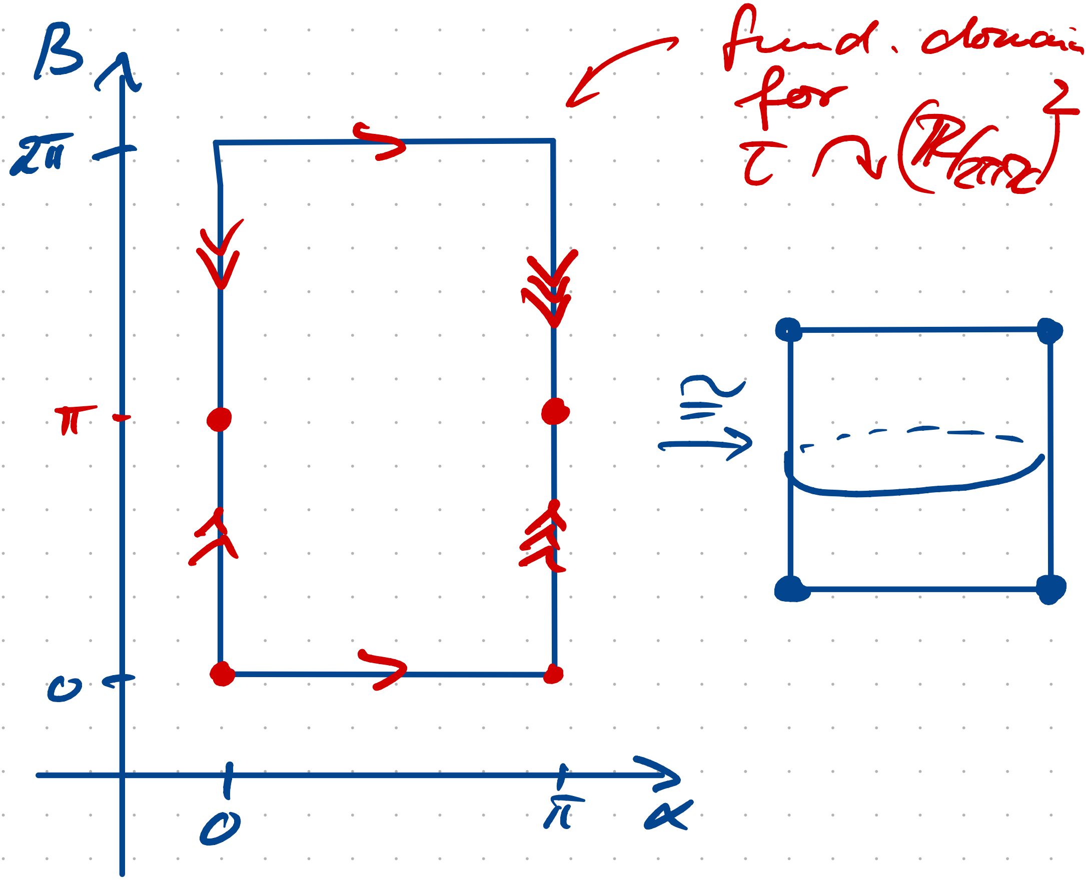
$$\rho(\ell) = \begin{pmatrix} e^{i\beta} & 0 \\ 0 & e^{-i\beta} \end{pmatrix} \qquad \beta \in \mathbb{R}/2\pi\mathbb{Z}$$

(α, β) are well-def. by $[\rho]$ only up to

$$\tau: (\alpha, \beta) \longmapsto (-\alpha, -\beta)$$
$$= (2\pi - \alpha, 2\pi - \beta)$$

$\alpha = 0$ $\beta \sim 2\pi - \beta$

$\alpha = \pi$ $\beta \sim 2\pi - \beta$



<u>Next:</u> $M = S^3 \setminus N(K) =: E(K)$

$$T^2 = \partial M \overset{i}{\hookrightarrow} M^3$$

induces

$$R(T^2) \xleftarrow{\ i^*\ } R(M)$$

Examples:

$\pi_1(E(K)) = \langle m \rangle$

$K = O$

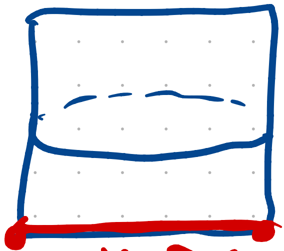
$R(E(K)) =: R(K)$

$\cong$

$SU(2)/SU(2)$

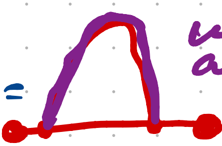
$\cong$

$\xrightarrow{i^*}$

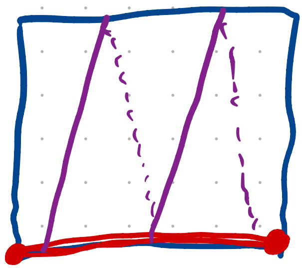


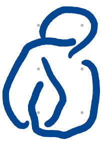
$i^* R(K)$

$K =$ 

$R(K) =$  non-abel.

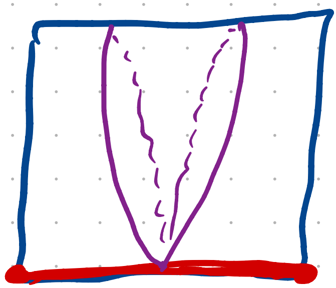
$\xrightarrow{i^*}$

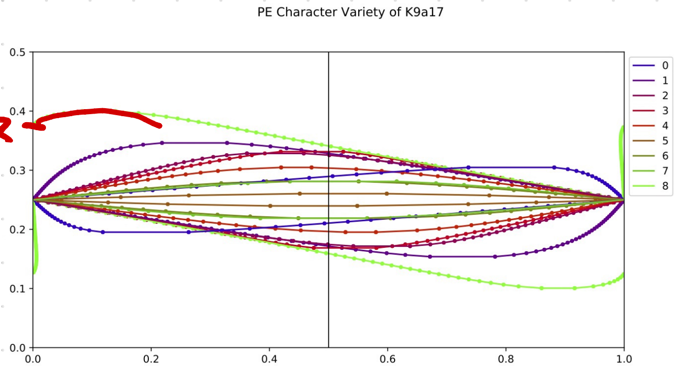


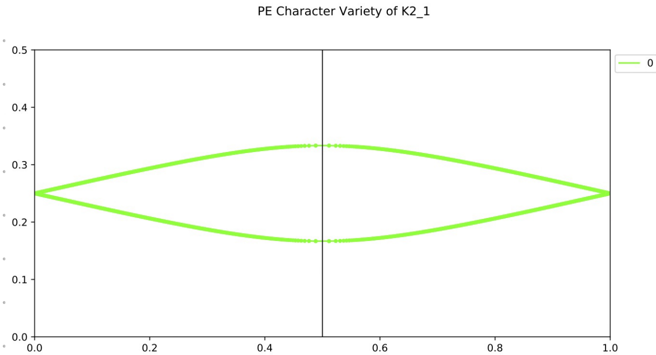
$K =$ 

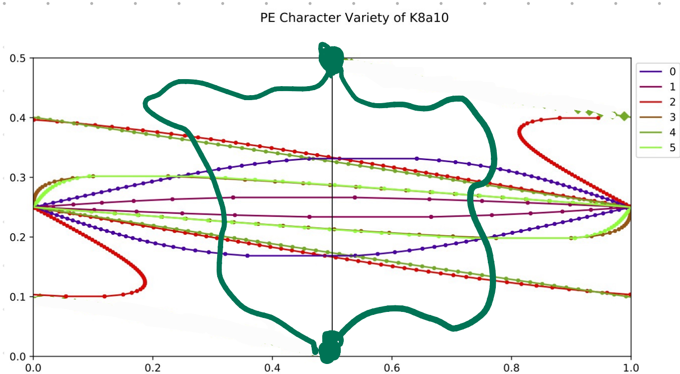
$R(K) =$  non-abel.

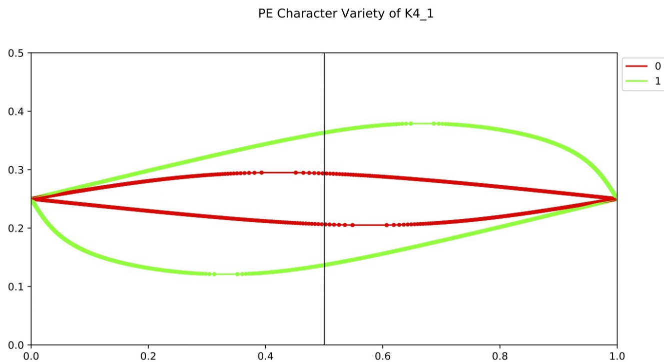
$\xrightarrow{i^*}$




PE Character Variety of K9a17
0.0 0.1 0.2 0.3 0.4 0.5
0.0 0.2 0.4 0.6 0.8 1.0
0 1 2 3 4 5 6 7 8


PE Character Variety of K2_1
0.0 0.1 0.2 0.3 0.4 0.5
0.0 0.2 0.4 0.6 0.8 1.0
0


PE Character Variety of K8a10
0.0 0.1 0.2 0.3 0.4 0.5
0.0 0.2 0.4 0.6 0.8 1.0
0 1 2 3 4 5


PE Character Variety of K4_1
0.0 0.1 0.2 0.3 0.4 0.5
0.0 0.2 0.4 0.6 0.8 1.0
0 1

Thm Z'16:

If $K \neq$ unknot, $K \subseteq S^3$, then $i^* R(K)$ „wraps around the pillowcase"

Thm (Lidman, Pinzon Caicedo, Z'16)

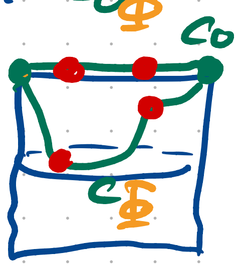
True also for knots $K \subseteq Y$ if Y has no non-trivial $SU(2)$-rep's. s.th. $Y \setminus N(K)$ is ∂-incompressible.

Pf:

Later: $I^w(Y_0(K)) \neq 0$

$R^w_\Phi(Y_0(K)) \cong i^* R(K) \cap C_\Phi$

$w \neq 0$



$I^w_\Phi(Y_0(K)) \neq 0$

$\Longrightarrow R^w(Y_0(K)) \neq \emptyset$

↑ holonomy perturbation

Abundance of Φ's

$\Longrightarrow$ True for any C $\square$

Thm (L-PC-Z).

$$I^w(Y_0(K)) \neq 0$$

Y has no irred. $su(2)$-rep's

Pf:

Thm (Kr + Mr) =

[If $b_1(M) = 1$, M is irred.

$\Longrightarrow I^w(M) \neq 0$.]

We don't know whether $Y_0(K)$ is irred.

$$I^w(Y_0(K)) \longrightarrow I(Y_{1/n}(K))$$

$$I(Y_{1/n}(K)) \longrightarrow I(Y_{1/n+1}(K)) \longrightarrow I^w(Y_0(K))$$

$\forall n \in \mathbb{Z}$ ($n = 0$ included)

Suppose $I^{\omega}(Y_0(K)) = 0$.

Hypothesis that Y has no $SU(2)$-rep's $\Rightarrow I(Y) = 0$

$\Rightarrow I(Y_1(K)) = 0$

induct.

$\Rightarrow I(Y_{\frac{1}{n}}(K)) = 0$

$\forall n \geq 0$

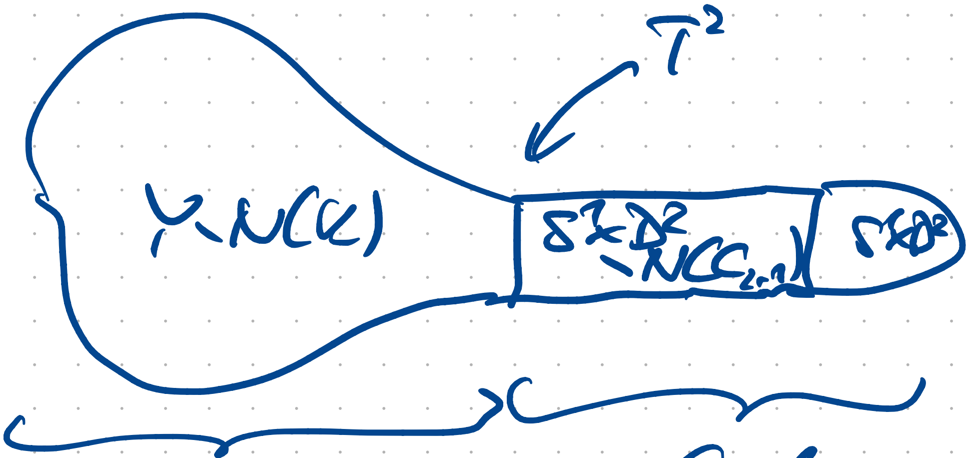
$$Y_{1/4}(K) \overset{\text{Gordon}}{=} Y_1(C_{2,1}(K))$$

$(2,1)$-cable

$$\Rightarrow I(Y_1(C_{2,1}(K))) = 0$$

$$\Rightarrow I^{\omega}(Y_0(C_{2,1}(K))) = 0.$$

$Y_0(C_{2,1}(K))$ is irred.



∂-incompressible

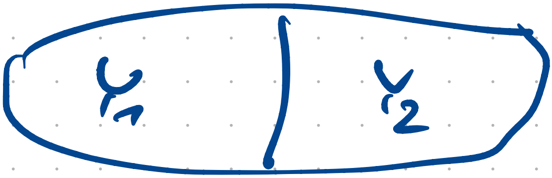
Gordon: is Seifert fibred with incompress ∂.

$\lightning$

Proof of Thm

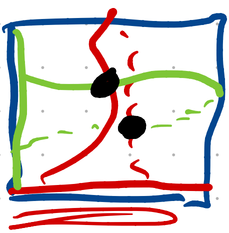
$\exists$ $SU(2)$-rep's for Y toroidal $\mathbb{Z}HS^3$

- $\exists\, T^2 \hookrightarrow Y$

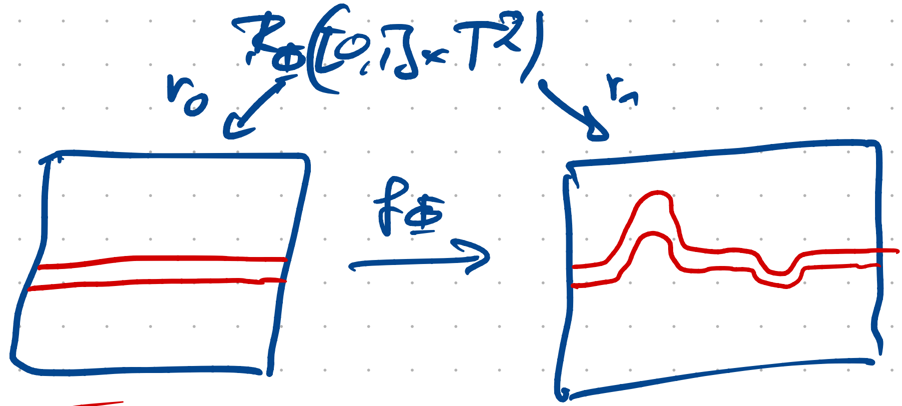
$Y =$ 

splicing of knots $K_1 \subseteq Y_1$, $K_2 \subseteq Y_2$

If both Y_1, Y_2 have no irred $SU(2)$-rep's

$R(K_1) \subseteq Y_1 \xrightarrow{i^*}$  $\xleftarrow{i^*} R(K_2)$

$\square$



$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \longmapsto \begin{pmatrix}\alpha\\ \beta + \underline{g}(\alpha)\end{pmatrix}$$